File S1: Supporting Information

Covariance properties of the fitness distribution

To analyze the covariances of different fitness classes, we begin with Eq. (7) of the main text, which expresses $\delta x_k(\tau)$ in terms of eigenvectors $\delta x_k(\tau) = \sum_j \psi_k^{(j)} a_j(\tau)$. Projecting on the left eigenvectors then results in equations for $a_j(\tau)$

$$\frac{d}{d\tau} a_j(\tau) = -j a_j(\tau) + \sum_k \phi_k^{(j)} \sqrt{\frac{\bar{x}_k}{Ns}} \eta_k(\tau) \tag{1}$$

where $\eta_k(\tau)$ are uncorrelated Gaussian white noise terms with $\langle \eta_k(\tau)\eta_l(\tau')\rangle = \delta_{kl}\delta(\tau - \tau')$. Since each noise term η_k contributes to all a_j, the noise induces correlated fluctuations of the $a_j(\tau)$, which we need to understand in order to analyze the fluctuations of the fitness distributions. The inhomogeneous Eq. (**??**) has the solution

$$a_j(\tau) = \int_{-\infty}^{\tau} d\tau' e^{-j(\tau-\tau')} \sum_k \phi_k^{(j)} \sqrt{\frac{\bar{x}_k}{Ns}} \eta_k(\tau') \tag{2}$$

The autocorrelation function of the loadings of different eigendirections separated by $\Delta\tau$ in time is therefore given by

$$\begin{aligned}
\langle a_i(\tau) a_j(\tau+\Delta\tau)\rangle &= \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau+\Delta\tau} d\tau'' e^{-i(\tau-\tau')-j(\tau+\Delta\tau-\tau'')} \sum_{k,l} \frac{\phi_k^{(i)}\phi_l^{(j)}\sqrt{\bar{x}_k \bar{x}_l}}{Ns} \langle \eta_k(\tau')\eta_l(\tau'')\rangle \\
&= \int_{-\infty}^{\tau} d\tau' e^{-i(\tau-\tau')-j(\tau+\Delta\tau-\tau')} \sum_k \frac{\phi_k^{(i)}\phi_k^{(j)}\bar{x}_k}{Ns} \\
&= \frac{e^{-j\Delta\tau}}{i+j} \sum_k \frac{\phi_k^{(i)}\phi_k^{(j)}\bar{x}_k}{Ns}
\end{aligned} \tag{3}$$

where we have used $\langle \eta_k(\tau)\eta_l(\tau')\rangle = \delta_{kl}\delta(\tau - \tau')$.

Correlation functions n_0 and the mean

To calculate the variances and covariance of x_0 and the mean fitness, we express them in terms of the eigenmodes $a_j(\tau)$ $(j>0)$

$$\delta x_0(\tau) = \sum_{j>0} \psi_0^{(j)} a_j(\tau) = e^{-\lambda} \sum_{j>0} a_j(\tau) \tag{4}$$

$$\delta \bar{k}(\tau) = \sum_{j>0,k} k\psi_k^{(j)} a_j(\tau) = \sum_{j>0,k} k(\bar{x}_{k-j} - \bar{x}_k) a_j(\tau) = \sum_{j>0} j a_j(\tau) \tag{5}$$

The auto-correlation of x_0

The auto-correlation of x_0 is given by

$$\begin{aligned}\langle x_0(\tau)x_0(\tau+\Delta\tau)\rangle &= e^{-2\lambda} \sum_{i,j>0} \frac{e^{-j\Delta\tau}}{i+j} \sum_k \frac{\phi_k^{(i)}\phi_k^{(j)}\bar{x}_k}{Ns} \\ &= \frac{e^{-\lambda}}{Ns} \sum_{i,j>0} \frac{\lambda^{i+j}e^{-j\Delta\tau}}{i+j} \sum_{k=0}^{\min(i,j)} \frac{(-1)^{i+j}\lambda^{-k}}{(j-k)!(i-k)!k!} \\ &= \frac{e^{-\lambda}}{Ns} \int_0^\infty dz \sum_{i,j>0} e^{-z(i+j)}\lambda^{i+j}e^{-j\Delta\tau} \sum_{k=0}^{\min(i,j)} \frac{(-1)^{i+j}\lambda^{-k}}{(j-k)!(i-k)!k!}\end{aligned} \tag{6}$$

Let us focus on the triple sum inside the integral and simplify it by introducing $a = -\lambda e^{-z}$ and $b = -\lambda e^{-z-\Delta\tau}$. Furthermore, let us look at the $i=j$ and the $i \neq j$ contributions separately. The diagonal contribution $(i=j)$ is

$$\begin{aligned}&\sum_{i=0} a^i b^i \sum_{k=0}^{i} \frac{\lambda^{-k}}{(i-k)!(i-k)!k!} = \sum_{i>0}\sum_{k=0}^{i} \frac{(ab)^{i-k}(ab)^k\lambda^{-k}}{k!((i-k)!)^2} \\ &= \sum_{k>0}\sum_{i\geq k} \frac{(ab)^{i-k}(ab)^k\lambda^{-k}}{k!((i-k)!)^2} + \sum_{i>0} \frac{(ab)^i}{(i!)^2} \\ &= \sum_{k>0} \frac{(ab)^k\lambda^{-k}}{k!} \sum_{n\geq 0} \frac{(ab)^n}{(n!)^2} + J_0(-\iota 2\sqrt{ab}) - 1 \qquad \text{using} \quad n = i-k \\ &= (e^{ab/\lambda}-1)J_0(-\iota 2\sqrt{ab}) + J_0(-\iota 2\sqrt{ab}) - 1 \\ &= e^{ab/\lambda}J_0(-\iota 2\sqrt{ab}) - 1\end{aligned} \tag{7}$$

where $J_n(z)$ is the nth Bessel function of first kind, and $\iota = \sqrt{-1}$. When evaluating the off-diagonal contribution, we will encounter terms like

$$\sum_{k>0} \frac{(ab)^k}{k!(k+m)!} = \sum_k \frac{(ab)^k}{k!(k+m)!} - \frac{1}{m!} = \frac{J_m(2\iota\sqrt{ab})}{(\iota\sqrt{ab})^m} - \frac{1}{m!} \tag{8}$$

The off-diagonal contribution can be further split into the parts $i > j$ and $i < j$ which can be evaluated as follows:

$$\begin{aligned}
&\sum_{0<i<j} a^i b^j \sum_{k\geq 0}^{i} \frac{\lambda^{-k}}{k!(j-k)!(i-k)!} = \sum_{i>0}\sum_{j>i} a^i b^{j-i+i} \sum_{k\geq 0}^{i} \frac{\lambda^{-k}}{k!(i+(j-i)-k)!(i-k)!} \\
&= \sum_{i>0}\sum_{m>0} a^i b^{m+i} \sum_{k\geq 0}^{i} \frac{\lambda^{-k}}{k!(i+m-k)!(i-k)!} \qquad \text{using} \quad m = j-i \\
&= \sum_{k>0}\sum_{m>0}\sum_{i\geq k} \frac{a^i b^{m+i}\lambda^{-k}}{k!(i+m-k)!(i-k)!} + \sum_{m>0}\sum_{i>0} \frac{a^i b^{m+i}}{(i+m)!i!} \\
&= \sum_{k>0}\sum_{m>0}\sum_{n\geq 0} \frac{a^{n+k} b^{m+n+k}\lambda^{-k}}{k!(n+m)!n!} + \sum_{m>0} b^m \sum_{i>0} \frac{(ab)^i}{(i+m)!i!} \qquad \text{using} \quad n = i-k \\
&= \sum_{k>0}\sum_{m>0} \frac{a^k b^{m+k}\lambda^{-k}}{k!} \frac{J_m(2\iota\sqrt{ab})}{(\iota\sqrt{ab})^m} + \sum_{m>0} b^m \left(\frac{J_m(2\iota\sqrt{ab})}{(\iota\sqrt{ab})^m} - \frac{1}{m!}\right) \\
&= \sum_{m>0}\sum_{k\geq 0} \frac{a^k b^k b^{m/2} a^{-m/2}\lambda^{-k}}{k!} \frac{J_m(2\iota\sqrt{ab})}{(\iota)^m} - e^b + 1 = \sum_{m>0} \left(\frac{b}{a}\right)^{m/2} e^{ab/\lambda} \frac{J_m(2\iota\sqrt{ab})}{(-1)^{m/2}} - e^b + 1 \\
&= e^{ab/\lambda} \sum_{m>0} \left(\frac{b}{a}\right)^{m/2} (-\iota)^m J_m(2\iota\sqrt{ab}) - e^b + 1
\end{aligned} \tag{9}$$

The off-diagonal terms for $i > j$ is obtained by interchanging a and b such that the full off-diagonal contribution is

$$e^{ab/\lambda} \sum_{m>0} \left[\left(\frac{b}{a}\right)^{m/2} + \left(\frac{a}{b}\right)^{m/2}\right] (-\iota)^m J_m(2\iota\sqrt{ab}) - e^a - e^b + 2 \tag{10}$$

Next, we use the definition of the generating function of the Bessel functions (**?**], 8.511)

$$e^{\frac{1}{2}(t-t^{-1})z} = J_0(z) + \sum_{m>0} (t^m + (-t)^{-m}) J_m(z) \tag{11}$$

which turns the off-diagonal contribution into

$$e^{ab/\lambda}\left(e^{\sqrt{ab}\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}\right)} - J_0(2\iota\sqrt{ab})\right) - e^a - e^b + 2 \tag{12}$$

Combining the diagonal and off-diagonal contributions and substituting a and b, we find for the integrand in Eq. (**??**)

$$e^{ab/\lambda + \sqrt{ab}\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}\right)} - e^a - e^b + 1 = e^{\lambda e^{-2z-\Delta\tau} - \lambda e^{-z} - \lambda e^{-z-\Delta\tau}} - e^{-\lambda e^{-z}} - e^{-\lambda e^{-z-\Delta\tau}} + 1 \tag{13}$$

The auto-correlation of x_0 is therefore given by

$$\begin{aligned}\langle x_0(\tau)x_0(\tau+\Delta\tau)\rangle &= \frac{e^{-\lambda}}{Ns}\int_0^\infty dz\left(e^{\lambda e^{-2z-\Delta\tau}}e^{-\lambda(e^{-z}+e^{-z-\Delta\tau})}-e^{-\lambda e^{-z}}-e^{-\lambda e^{-z-\Delta\tau}}+1\right)\\ &= \frac{e^{-\lambda}}{Ns}\int_0^1 \frac{d\theta}{\theta}\left(e^{\lambda\theta^2e^{-\Delta\tau}-\lambda\theta(1+e^{-\Delta\tau})}-e^{-\lambda\theta}-e^{-\lambda\theta e^{-\Delta\tau}}+1\right)\end{aligned} \tag{14}$$

Auto-correlation of the mean fitness

The autocorrelation function of the mean is defined as

$$\begin{aligned}\langle\delta\bar{k}(\tau)\delta\bar{k}(\tau+\Delta\tau)\rangle &= \sum_{i,j>0} ij\langle a_i(\tau)a_j(\tau+\Delta\tau)\rangle\\ &= \partial_\mu\partial_\nu\frac{1}{Ns}\int_0^\infty dz\sum_{i,j>0}\mu^i\nu^j e^{-z(i+j)}\lambda^{i+j}e^{-j\Delta\tau}\sum_{k=0}^{\min(i,j)}\frac{(-1)^{i+j}\lambda^{-k}}{(j-k)!(i-k)!k!}\end{aligned} \tag{15}$$

where the last line is to be evaluated at $\nu=\mu=1$. Hence the problem is reduced to the one already solved with $a=-\mu\lambda e^{-z}$ and $b=-\nu\lambda e^{-z-\Delta\tau}$. We find

$$\langle\delta\bar{k}(\tau)\delta\bar{k}(\tau+\Delta\tau)\rangle = \frac{\lambda e^{\lambda}}{Ns}\int_0^1 d\theta e^{-\Delta\tau}e^{\lambda\theta^2e^{-\Delta\tau}-\lambda(1+e^{-\Delta\tau})\theta}\left(\theta+\lambda\theta\left(\theta e^{-\Delta\tau}-1\right)(\theta-1)\right) \tag{16}$$

Cross-correlation of x_0 and the mean fitness

When calculating the cross-correlation between x_0 and the mean fitness we have to distinguish the cases where x_0 precedes the mean fitness and vice-versa. Otherwise, the calculation proceeds almost unchanged from the cases discussed above.

$$\begin{aligned}\langle x_0(\tau)\delta\bar{k}(\tau+\Delta\tau)\rangle &= \sum_{i,j>0} j\langle a_i(\tau)a_j(\tau+\Delta\tau)\rangle\\ &= \partial_\nu\frac{1}{Ns}\int_0^\infty dz\sum_{i,j>0}a^ib^j\sum_{k=0}^{\min(i,j)}\frac{(-1)^{i+j}\lambda^{-k}}{(j-k)!(i-k)!k!}\end{aligned} \tag{17}$$

with $a=-\lambda e^{-z}$, $b=-\nu\lambda e^{-z-\Delta\tau}$ if $\Delta\tau>0$ and $a=-\lambda e^{-z+\Delta\tau}$, $b=-\nu\lambda e^{-z}$ if $\Delta\tau<0$. The result is

$$\langle\delta x_0(\tau)\delta\bar{k}(\tau+\Delta\tau)\rangle = \frac{\lambda}{Ns}\begin{cases} e^{-\Delta\tau}\int_0^1 d\theta\left((\theta-1)e^{e^{-\Delta\tau}\lambda\theta^2-\lambda(1+e^{-\Delta\tau})\theta}+e^{-e^{-\Delta\tau}\lambda\theta}\right) & \Delta\tau>0\\ \int_0^1 d\theta\left((e^{\Delta\tau}\theta-1)e^{e^{\Delta\tau}\lambda\theta^2-\lambda(1+e^{\Delta\tau})\theta}+e^{-\lambda\theta}\right) & \Delta\tau<0\end{cases} \tag{18}$$

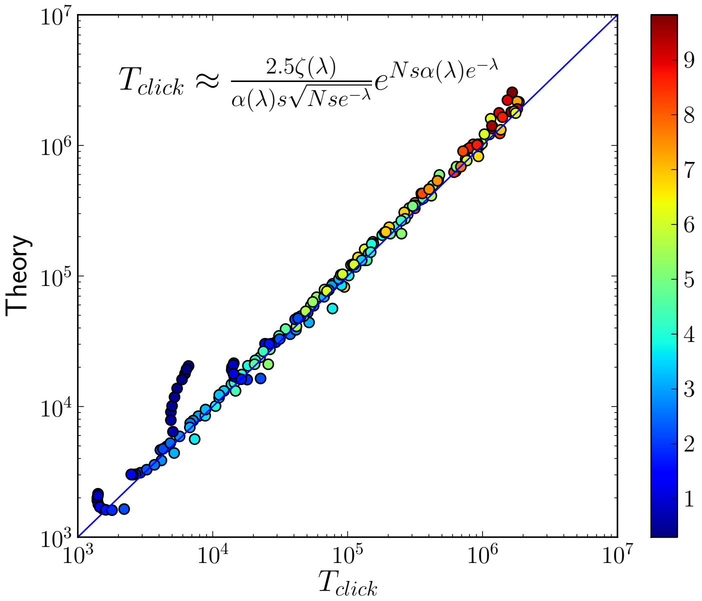

Figure S 1: The approximation of the mean time between clicks of the ratchet is accurate over a large range of parameters if $Ns\alpha(\lambda)e^{-\lambda}$ is large compared to one. $Ns\alpha(\lambda)e^{-\lambda}$ determines whether the clicks of the ratchet are far apart compared to the relaxation time of the distribution and is indicated as the color of the data points. The condition $Ns\alpha(\lambda)e^{-\lambda} > 1$ is violated for the fastest clicks shown, resulting in the deviation of the dark blue points.