File S1
Supporting Information

PROOF OF EQUIVALENCE OF $\hat{\pi}_{\texttt{SMC}}$ AND $\hat{\pi}_{\texttt{PS,1}}$

We now want to give a more detailed proof of Proposition 1 from the main text. With the notation as in the paper we have:

**Proposition 1.** For an arbitrary single haplotype $\alpha \in \mathcal{H}$ and haplotype configuration $\mathbf{n}$, $\hat{\pi}_{\texttt{SMC}}(\alpha | \mathbf{n}) = \hat{\pi}_{\texttt{PS,1}}(\alpha | \mathbf{n})$.

Recall that the initial (stationary) density was given by

$$\zeta^{(\mathbf{n})}(S_\ell = (t_\ell, h_\ell)) = \frac{n h_\ell}{2} e^{-\frac{n}{2} t_\ell}, \quad \text{(S.1)}$$

the transition density by

$$\phi^{(\mathbf{n})}_{\mathbf{p}_\beta}(s_\ell | s_{\ell-1}) = e^{-\frac{\psi}{2} t_\ell-1} \delta_{s_\ell-1, s_\ell} + \frac{n a_{t_\ell}}{n} \frac{\beta h_\ell}{2} e^{-\frac{n}{2} t_\ell} e^{-\frac{n}{2} (t_\ell - t_\beta)}, \quad \text{(S.2)}$$

and the emission probability by

$$\xi^{(\mathbf{n})}_{\theta_\ell}(\alpha[\ell] | s_\ell) = \left[ e^{-\frac{\psi}{2} t_\ell} P(\ell^\ell - 1) \right]_{h_\ell[\ell], \alpha[\ell]}, \quad \text{(S.3)}$$

Since the configuration $\mathbf{n}$ is fixed, we will drop the superscript $(\mathbf{n})$ in the sequel. As in the main text we will also omit the recombination and mutation rate when unambiguous. Further, we will omit the $d$-whenever we write down integrals. If not specified differently, equation-references refer to equations from the main paper.

**Proof of Proposition 1.** We start by showing inductively that the joint density $f_{\texttt{SMC}}(\alpha[\ell' : \ell], (t_\ell, h_\ell))$ of observing the partial haplotype $\alpha[\ell' : \ell]$ and being in the hidden state $(t_\ell, h_\ell)$ (basically) introduced in equations (4)-(6) satisfies a genealogical recursion $f$, defined as follows [c.f., Griffiths and Tavaré (1994)]:

$$f(\alpha[\ell' : \ell], (t_\ell, h_\ell)) = \int_{t_p=0}^{t_\ell} e^{-n + \sum_u \theta_u + \sum_u \rho_u \int t_p} \left[ \frac{n h_\ell \delta_{\alpha[\ell' : \ell], h_\ell[\ell' : \ell]}}{2} \delta_{t_p, t_\ell} + \sum_{u \in L(\ell' : \ell)} \delta_{u} \sum_{a \in E_a} P^{(u)}_{a, \alpha[u]} f(S^{(u)}_a(\alpha)[\ell' : \ell], (t_\ell - t_p, h_\ell)) + \sum_{u \in B(\ell' : \ell)} \rho_u \left( \int_{s_u} f(\alpha[\ell' : u_1], s_{u_1}) f(\alpha[u_2, \ell], (t_\ell - t_p, h_\ell)) \right) \right], \quad \text{(S.4)}$$

where the sum $\sum_u$ is over $L(\ell' : \ell)$, all loci between (and including) $\ell'$ and $\ell$, and $\sum_{a}$ is over $B(\ell' : \ell)$, the breakpoints between $\ell'$ and $\ell$. Here $S^{(u)}_a(\alpha)$ denotes the haplotype obtained by substituting the allele $a$ at locus $u$ of $\alpha$, and $u = (u_1, u_2)$. For $\ell' = \ell$ (so that $\ell - \ell' = 0$),

$$f(\alpha[\ell], s_\ell) = \int_{t_p=0}^{t_\ell} e^{-n + \theta_p \int t_p} \left[ \frac{n h_\ell \delta_{\alpha[\ell], h_\ell[\ell]}}{2} \delta_{t_p, t_\ell} + \frac{\theta}{2} \sum_{a \in E_\ell} P^{(\ell)}_{a, \alpha[\ell]} f(a, (t_\ell - t_p, h_\ell)) \right].$$
Substituting $f = f_{\text{SMC}}$ on the right-hand side,

$$
\int_{t_p=0}^{t_{\ell}} e^{-\frac{n+\eta}{2}t_p} \left[ \frac{n_{h_\ell} \delta_{\alpha}[\ell], h_\ell[\ell]}{2} \delta_{t_p, t_{\ell}} + \frac{\theta_\ell}{2} \sum_{a \in \mathcal{E}_\ell} P_{a, \alpha[\ell]}^{(\ell)} f_{\text{SMC}}(a, (t_{\ell} - t_p, h_\ell)) \right]
\]

$$

$$
= e^{-\frac{n+\eta}{2}t_{\ell}} n_{h_\ell} \delta_{\alpha}[\ell], h_\ell[\ell] + \sum_{m=0}^{\infty} \left( \sum_{a \in \mathcal{E}_\ell} P_{a, \alpha[\ell]}^{(\ell)} [\delta_{\alpha}[\ell], h_\ell[\ell], a] \right) \int_{t_p=0}^{t_{\ell}} \frac{\theta_\ell}{2} \left( \frac{\theta_\ell}{2} \right)^m \frac{1}{m!}
\]

$$

$$
= e^{-\frac{n+\eta}{2}t_{\ell}} \left( \delta_{\alpha}[\ell], h_\ell[\ell] + \sum_{m=0}^{\infty} \left[ (\delta_{\alpha}[\ell], h_\ell[\ell], a] \right) \left( \frac{\theta_\ell}{2} \right)^m \frac{1}{m!} \right)
\]

$$

$$
= e^{-\frac{n+\eta}{2}t_{\ell}} \left[ e^{\frac{\theta_\ell}{2} t_{\ell} P_{(\ell)} - f_\ell} \right]_{h_\ell[\ell], \alpha}[\ell],
\]

with the final result equal to $\xi(\alpha[\ell]) \mid s_{\ell}) = f_{\text{SMC}}(\alpha[\ell], s_{\ell})$. Now, inductively assuming that $f_{\text{SMC}}(\alpha[\ell'] : s_{\ell}) = f(\alpha[\ell'] : s_{\ell})$ for $0 \leq \ell - \ell' < m$ and all values of $s_{\ell}$, let $\ell' < \ell$ such that $\ell - \ell' = m$.

Substituting $f = f_{\text{SMC}}$ on the right-hand side of (S.4), we obtain

$$
\int_{t_p=0}^{t_{\ell}} e^{-\frac{n+\eta}{2}t_p} \left[ \frac{n_{h_\ell} \delta_{\alpha}[\ell'], h_\ell[\ell']}{2} \delta_{t_p, t_{\ell}} + \sum_{u \in \mathcal{L}(\ell', \ell)} \frac{\theta_u}{2} \sum_{a \in \mathcal{E}_u} P_{a, \alpha[u]}^{(u)} f_{\text{SMC}}(\mathcal{S}_u^a(\alpha)[\ell'] : (t_{\ell} - t_p, h_\ell))
\]

$$

$$
+ \sum_{u \in \mathcal{B}(\ell' : \ell)} \frac{\rho_u}{2} \left( \int_{s_{u1}} f_{\text{SMC}}(\alpha[\ell' : u_1], s_{u1}) \right) f_{\text{SMC}}(\alpha[\ell' : u_1], (t_{\ell} - t_p, h_\ell)) \right]
\]

We consider this expression one term at a time. Beginning with the first term:

$$
\int_{t_p=0}^{t_{\ell}} e^{-\frac{n+\eta}{2}t_p} \left[ \frac{n_{h_{\ell-1}} \delta_{\alpha}[\ell', t_{\ell-1}], h_{\ell-1}[\ell', t_{\ell-1}]}{2} \delta_{t_p, t_{\ell-1}} + \sum_{u \in \mathcal{L}(\ell': \ell-1)} \frac{\theta_u}{2} \sum_{a \in \mathcal{E}_u} P_{a, \alpha[u]}^{(u)} f_{\text{SMC}}(\mathcal{S}_u^a(\alpha)[\ell'] : (t_{\ell} - t_p, h_\ell))
\]

$$

$$
= \int_{s_{\ell-1}}^{t_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\eta}{2}t_p} n_{h_{\ell-1}} \delta_{\alpha}[\ell', t_{\ell-1}], h_{\ell-1}[\ell', t_{\ell-1}] \delta_{t_p, t_{\ell-1}} \left[ e^{-\frac{\theta_{\ell-1} p_{\ell' \ell-1}}{2} t_{\ell} P_{(\ell')} + f_{\ell}} \right]
\]

$$

where $\sum'_u$ is over $L(\ell' : \ell - 1)$ and $\sum'_u$ is over $B(\ell' : \ell - 1)$. Moving on to the second term, expand using the definition (5) of $f_{\text{SMC}}$ and then use the inductive hypothesis to replace the resulting $f_{\text{SMC}}$ terms with the corresponding $f$ terms:
Concentrating on the first sub-term, making the substitution \( t_{\ell-1} \rightarrow t_{\ell-1} + t_p \), and changing the order of integration, we obtain

\[
\int_{s_{\ell-1}}^{t_{\ell-1}} e^{-\frac{\gamma u}{2} t_p} \prod_{u \in L(\ell' : \ell - 1)} \frac{n_u}{2} \sum_{a \in E_u} P_{a,\alpha[u]} \xi(a | (t_{\ell-1} - t_p, h_{\ell-1})) \int_{s_{\ell-1}}^{t_{\ell-1}} \phi((t_{\ell-1} - t_p, h_{\ell-1})) [ (t_{\ell-1} - t_p, h_{\ell-1}) ]
\]

Now concentrating on the second sub-term and expanding using definition (S.4) of \( f \):

\[
\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{\gamma u}{2} t_p} \prod_{u \in L(\ell' : \ell - 1)} \frac{n_u}{2} \sum_{a \in E_u} P_{a,\alpha[u]} \xi(a | (t_{\ell-1} - t_p, h_{\ell-1})) \int_{s_{\ell-1}}^{t_{\ell-1}} \phi((t_{\ell-1} - t_p, h_{\ell-1})) [ (t_{\ell-1} - t_p, h_{\ell-1}) ]
\]

with the equality obtained by making the substitutions \( t_{\ell-1} \rightarrow t_{\ell-1} + t_p \) and \( t_q \rightarrow t_q + t_p \) and then changing the order of integration. Finally, moving onto the third term, expand using the definition (5) of \( f_{\text{SMC}} \), and then use the inductive hypothesis to replace the resulting \( f_{\text{SMC}} \) terms with the corresponding \( f \) terms:

\[
\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{\gamma u}{2} t_p} \prod_{u \in L(\ell' : \ell - 1)} \frac{n_u}{2} \sum_{a \in E_u} P_{a,\alpha[u]} \xi(a | (t_{\ell-1} - t_p, h_{\ell-1})) \int_{s_{\ell-1}}^{t_{\ell-1}} \phi((t_{\ell-1} - t_p, h_{\ell-1})) [ (t_{\ell-1} - t_p, h_{\ell-1}) ]
\]
Concentrating on the first sub-term, making the substitution \( t_{\ell-1} \to t_{\ell-1} + t_p \), and changing the order of integration, we obtain:

\[
\int_{t_p=0}^{t_\ell} \int_{s_{\ell-1}}^{t_{\ell} \land t_{\ell-1}} e^{-\frac{\mathbf{n} \cdot \mathbf{\theta} + \sum \mathbf{n} \cdot \mathbf{\mu}}{2}} t_p \sum_{u \in B(l', \ell - 1)} \frac{\rho_u}{2} \left( \int_{s_{u1}} f(\alpha[l', u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1})) \\
\times \left[ e^{-\frac{\mathbf{n} \cdot \mathbf{\phi}}{2}} \xi(\alpha[l] | (t_\ell - t_p, h_\ell)) \cdot e^{-\frac{\mathbf{n} \cdot \mathbf{\mu}}{2}} \phi((t_\ell - t_p, h_\ell) | (t_{\ell-1} - t_p, h_{\ell-1})) \right].
\]  

(S.9)

Now concentrating on the second sub-term and expanding using definition (S.4) of \( f \):

\[
\int_{t_p=0}^{t_\ell} e^{-\frac{\mathbf{n} \cdot \mathbf{\theta} + \sum \mathbf{n} \cdot \mathbf{\mu}}{2}} t_p \frac{\rho_u}{2} f(\alpha[l], (t_\ell - t_p, h_\ell)) \\
\times \int_{s_{\ell-1}}^{t_{\ell-1}} e^{-\frac{\mathbf{n} \cdot \mathbf{\theta} + \sum \mathbf{n} \cdot \mathbf{\mu}}{2}} t_q \left[ n_{h_{\ell-1}} \delta_\alpha[l', \ell - 1], h_{\ell-1}[l', \ell - 1] \right] \frac{\rho_u}{2} \left( \int_{s_{u1}} f(\alpha[l', u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1})) \\
+ \sum_{u \in L(l', \ell - 1)} \frac{\theta_u}{2} \sum_{a \in E_u} f(\mathcal{S}_a^u(\alpha)[l' : \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1})) \\
+ \sum_{u \in B(l', \ell - 1)} \frac{\rho_u}{2} \left( \int_{s_{u1}} f(\alpha[l', u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1})) \\
\times \left[ \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{\mathbf{n} \cdot \mathbf{\phi}}{2}} \xi(\alpha[l] | (t_\ell - t_p, h_\ell)) \cdot e^{-\frac{\mathbf{n} \cdot \mathbf{\mu}}{2}} \phi((t_\ell - t_p, h_\ell) | (t_{\ell-1} - t_q, h_{\ell-1})) \right].
\]  

(S.10)

with the equality obtained by using the (one-locus) definition (6) for \( f_{\text{SMC}}(\alpha[l], (t_\ell - t_p, h_\ell)) \), making the substitutions \( t_{\ell-1} \to t_{\ell-1} + t_p \) and \( t_q \to t_q + t_p \), and changing the order of integration.

Having appropriately expanded each term of our key expression (S.5), we aggregate common terms across the resulting sub-expressions. Collecting the \( n_{h_{\ell-1}} \delta_\alpha[l', \ell - 1], h_{\ell-1}[l', \ell - 1] \) terms from (S.6),(S.8),
and (S.10),

\[
\begin{align*}
&\int_{s_{\ell - 1}}^{t_{\ell - 1}} \int_{t_p = 0}^{t_{\ell - 1}} e^{-\frac{n + \sum_{u}^t \sum_{\nu}^t \nu_{tu}}{2}} n_{h_{\ell - 1}} \delta_{\alpha[\ell', \ell - 1], h_{\ell - 1}[\ell', \ell - 1]} \delta_{t_p, \ell - 1} \\
&\times \left[ e^{-\frac{\Theta_p + \rho_h}{2} t_p} \delta_{\alpha[\ell], h_{\ell}[\ell]} \delta_{s_{\ell - 1}, \ell} \\
&+ \int_{t_q = 0}^{t_{\ell - 1} \wedge t_{\ell'}} e^{-\frac{\Theta_p}{2} t_q} \frac{\Theta_p}{2} \sum_{a \in E_u} P_{a, \alpha[\ell]} \xi(\alpha | (t_{\ell} - t_q, h_{\ell})) \cdot e^{-\frac{\rho_h}{2} t_q} \phi((t_{\ell} - t_q, h_{\ell}) | (t_{\ell - 1} - t_q, h_{\ell - 1})) \\
&+ \int_{t_q = 0}^{t_{\ell - 1} \wedge t_{\ell'}} e^{-\frac{\rho_h}{2} t_q} \xi(\alpha[\ell]) | (t_{\ell} - t_q, h_{\ell})) \cdot e^{-\frac{\rho_h}{2} t_q} \rho_h n_{h_{\ell}} e^{-\frac{\Phi_h}{2} (t_{\ell} - t_q)} \right] \\
&= \int_{s_{\ell - 1}}^{t_{\ell - 1}} \int_{t_p = 0}^{t_{\ell - 1}} e^{-\frac{n + \sum_{u}^t \sum_{\nu}^t \nu_{tu}}{2}} n_{h_{\ell - 1}} \delta_{\alpha[\ell', \ell - 1], h_{\ell - 1}[\ell', \ell - 1]} \delta_{t_p, \ell - 1} \\
&\times \left[ e^{-\frac{\rho_h}{2} t_{\ell - 1} \delta_{s_{\ell - 1}, \ell} \delta_{s_{\ell}} \left( \int_{t_z = 0}^{t_{\ell}} e^{-\frac{\Theta_p}{2} t_z} \frac{\Theta_p}{2} \sum_{a \in E_u} P_{a, \alpha[\ell]} \xi(\alpha | (t_{\ell} - t_z, h_{\ell})) \right) \\
&+ \int_{t_q = 0}^{t_{\ell - 1} \wedge t_{\ell'}} \rho_h e^{-\frac{\rho_h}{2} t_q} n_{h_{\ell}} e^{-\frac{\Phi_h}{2} (t_{\ell} - t_q)} \left( \int_{t_z = 0}^{t_q} e^{-\frac{\Theta_p}{2} t_z} \frac{\Theta_p}{2} \sum_{a \in E_u} P_{a, \alpha[\ell]} \xi(\alpha | (t_{\ell} - t_z, h_{\ell})) \right) \\
&+ \int_{t_q = 0}^{t_{\ell - 1} \wedge t_{\ell'}} \rho_h e^{-\frac{\rho_h}{2} t_q} n_{h_{\ell}} e^{-\frac{\Phi_h}{2} (t_{\ell} - t_q)} \left( e^{-\frac{\rho_h}{2} t_q} \xi(\alpha[\ell]) | (t_{\ell} - t_q, h_{\ell}) \right) \right] \\
&= \int_{s_{\ell - 1}}^{t_{\ell - 1}} \int_{t_p = 0}^{t_{\ell - 1}} e^{-\frac{n + \sum_{u}^t \sum_{\nu}^t \nu_{tu}}{2}} n_{h_{\ell - 1}} \delta_{\alpha[\ell', \ell - 1], h_{\ell - 1}[\ell', \ell - 1]} \delta_{t_p, \ell - 1} \\
&\times \xi(\alpha[\ell]) | s_{\ell} \left[ e^{-\frac{\rho_h}{2} t_{\ell - 1} \delta_{s_{\ell - 1}, \ell} \delta_{s_{\ell}} + \int_{t_q = 0}^{t_{\ell - 1} \wedge t_{\ell'}} \rho_h e^{-\frac{\rho_h}{2} t_q} n_{h_{\ell}} e^{-\frac{\Phi_h}{2} (t_{\ell} - t_q)} \right] \\
&= \int_{s_{\ell - 1}}^{t_{\ell - 1}} \int_{t_p = 0}^{t_{\ell - 1}} e^{-\frac{n + \sum_{u}^t \sum_{\nu}^t \nu_{tu}}{2}} n_{h_{\ell - 1}} \delta_{\alpha[\ell', \ell - 1], h_{\ell - 1}[\ell', \ell - 1]} \delta_{t_p, \ell - 1} \times \left[ \xi(\alpha[\ell]) | s_{\ell} \phi(s_{\ell} | s_{\ell - 1}) \right], \tag{S.11} \end{align*}
\]

where the first equality is obtained by making use of the $\delta_{t_{\ell}, \ell - 1}$ and $\delta_{s_{\ell - 1}, \ell}$ expressions and expanding the $q$ term using equation (S.2) and exchanging integrals, the second equality is obtained by combining the first/second and third/fourth term along with the definition (S.3) of $p$, and final equality by again making use of the equation (S.2).

Similarly, collecting the $f(S_u^a(\alpha)[\ell' \subseteq \ell - 1], (t_{\ell - 1} - t_q, h_{\ell - 1}))$ terms from the resulting sub-expressions
\[(S.7), (S.8), \text{and} (S.10),\]
\[
\int_{s_{\ell-1}}^{t_{\ell-1}} e^{-\frac{n\sum \theta u + \sum \rho u}{2} t_p} \sum_{u \in L(\ell', \ell-1)} \frac{\theta_u}{2} \sum_{a \in E_u} P_a^{(u)} f(S_u^a(\alpha)[\ell' : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1}))
\]
\[
\times \left[ \mathbb{I}(t_p \leq t_\ell) e^{-\frac{\rho u}{2} t_q} \xi(\alpha[\ell]) \mid (t_\ell - t_p, h_\ell) \mid (t_{\ell-1} - t_p, h_{\ell-1}) \right]
\]
\[
+ \int_{t_q = 0}^{t_{\ell-1}} e^{-\frac{\rho u}{2} t_q} \frac{\theta_u}{2} \sum_{a \in E_u} P_a^{(u)} f(S_u^a(\alpha)[\ell' : \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1}))
\]
\[
+ \int_{t_q = 0}^{t_{\ell-1}} e^{-\frac{\rho u}{2} t_q} \xi(\alpha[\ell]) \mid (t_\ell - t_q, h_\ell) \mid (t_{\ell-1} - t_q, h_{\ell-1}) \left[ \int_{t_q = 0}^{t_{\ell-1}} e^{-\frac{\rho u}{2} t_q} \frac{\theta_u}{2} \sum_{a \in E_u} P_a^{(u)} f(S_u^a(\alpha)[\ell' : \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1})) \right].
\]

where the first equality is obtained by expanding the \(\phi\) term\(^1\) in the second term using equation (S.2), the second equality is obtained by combining the first/second and third/fourth term along with the definition (S.3) of \(\xi\), and final equality by again making use of the equation (S.2) and considering separately the case when \(t_p \leq t_\ell\) and \(t_q > t_\ell\).

The situation is identical when collecting terms with \(f(\alpha[u_2, \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1}))\) from (S.9),

\(^1\)We use the following expansion for \(\phi\), which can be verified in the present context, namely that \(t_q \leq t_p \leq t_{\ell-1}\) and \(t_q \leq t_\ell\):

\[
\phi((t_\ell - t_q, h_\ell) \mid (t_{\ell-1} - t_q, h_{\ell-1})) = \mathbb{I}(t_p \leq t_\ell) e^{-\frac{\rho u}{2} (t_p - t_q)} \phi((t_\ell - t_p, h_\ell) \mid (t_{\ell-1} - t_p, h_{\ell-1}))
\]
\[
+ \int_{t_q = 0}^{t_{\ell-1}} e^{-\frac{\rho u}{2} t_q} \frac{\theta_u}{2} \sum_{a \in E_u} P_a^{(u)} f(S_u^a(\alpha)[\ell' : \ell - 1], (t_{\ell-1} - t_q, h_{\ell-1}))
\]
\[
\times \left[ \xi(\alpha[\ell]) \mid s_\ell \right].
\]

\[
(S.12)
\]
(S.8), and (S.10):

\[
\int_{s_{\ell-1}}^{t_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_{u}^{\prime} \theta_{u} + \sum_{u}^{\prime} \rho_{u}}{2}} \sum_{u \in B(\ell', \ell-1)} \frac{\rho_{u}}{2} \left( \int_{s_{u1}} f(\alpha[\ell': u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1})) \]

\[
\times \left[ \int_{t_p \leq t_{\ell}} e^{-\frac{\theta_{\ell}}{2} t_p} \xi(\alpha[\ell] | (t_{\ell} - t_q, h_{\ell})) \cdot e^{-\frac{\rho_{\ell}}{2} t_q} \phi((t_{\ell} - t_q, h_{\ell}) | (t_{\ell-1} - t_q, h_{\ell-1})) \right] \\
+ \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{\theta_{\ell}}{2} t_q} \sum_{a \in E_u} P_{a,\alpha[\ell]}(a | (t_{\ell} - t_q, h_{\ell})) \cdot e^{-\frac{\rho_{b}}{2} t_q} \frac{\rho_{b}}{2} n_{h_{\ell}} e^{-\frac{\sigma}{2} (t_{\ell} - t_q)} \\
+ \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{\theta_{\ell}}{2} t_q} \xi(\alpha[\ell] | (t_{\ell} - t_q, h_{\ell})) \cdot e^{-\frac{\rho_{\ell}}{2} t_q} \frac{\rho_{u}}{2} e^{-\frac{\sigma}{2} (t_{\ell} - t_q)} \\
= \int_{s_{\ell-1}}^{t_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_{u}^{\prime} \theta_{u} + \sum_{u}^{\prime} \rho_{u}}{2}} \sum_{u \in B(\ell', \ell-1)} \frac{\rho_{u}}{2} \left( \int_{s_{u1}} f(\alpha[\ell': u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1})) \]

\[
\times \left[ \xi(\alpha[\ell] | s_{\ell}) \phi(s_{\ell} | s_{\ell-1}) \right]. \tag{S.13}
\]

Thus, combining equations (S.11), (S.12), and (S.13), we may re-write (S.5):

\[
\xi(\alpha[\ell] | s_{\ell}) \int_{s_{\ell-1}} f(\alpha[\ell] | s_{\ell-1}) \cdot \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_{u}^{\prime} \theta_{u} + \sum_{u}^{\prime} \rho_{u}}{2}} \sum_{u \in B(\ell', \ell-1)} \frac{\rho_{u}}{2} \left( \int_{s_{u1}} f(\alpha[\ell': u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1})) \\
+ \sum_{u \in L(\ell', \ell-1)} \frac{\theta_{u}}{2} \sum_{a \in E_u} P_{a,\alpha[\ell]}(a | s_{u1}) f(S^a_u(\alpha)[\ell' : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1})) \\
+ \sum_{u \in B(\ell', \ell-1)} \frac{\rho_{u}}{2} \left( \int_{s_{u1}} f(\alpha[\ell': u_1], s_{u1}) \right) f(\alpha[u_2 : \ell - 1], (t_{\ell-1} - t_p, h_{\ell-1})) \\
= \xi(\alpha[\ell] | s_{\ell}) \int_{s_{\ell-1}} \phi(s_{\ell} | s_{\ell-1}) f(\alpha[\ell' : \ell - 1], s_{\ell-1}) \\
= f_{\text{SMC}}(\alpha[\ell' : \ell], s_{\ell}),
\]

where the first equality is obtained by definition (S.4) for $f$, and the second equality by using the inductive hypothesis and the definition (5). Therefore, $f_{\text{SMC}}$ satisfies the recursion for $f$, and we
conclude that \( f_{\text{SMC}} = f \). Moreover,
\[
\int_{s_{\ell}} f(\alpha[\ell': \ell], s_{\ell}) = \int_{s_{\ell}} \int_{t_p=0}^{t_{\ell}} e^{-\frac{n+\sum a_n+\sum a_n t_p}{2}} \left[ \frac{n h_{\ell} \delta_{\alpha[\ell': \ell], h_{\ell}[\ell', \ell]} \delta_{t_p, \ell}}{2} \right] \]
\[
+ \sum_{u \in L(\ell'; \ell)} \frac{\theta_u}{2} \sum_{a \in E_u} P_{a, \alpha[u]}^{(u)} f(S_{u}^{a}(\alpha)[\ell': \ell], (t_{\ell} - t_p + h_{\ell}))
\]
\[
+ \sum_{u \in B(\ell'; \ell)} \frac{\rho_u}{2} \left( \int_{s_{u_1}} f(\alpha[\ell': u_1], s_{u_1}) f(\alpha[u_2, \ell], (t_{\ell} - t_p + h_{\ell})) \right)
\]
\[
= \frac{1}{n + \sum_{u \in L(\alpha[\ell'; \ell])} \theta_u + \sum_{u \in B(\alpha[\ell'; \ell])} \rho_u} \left[ \sum_{\alpha' \in H} n_{\alpha'} \right]
\]
\[
+ \sum_{u \in L(\alpha[\ell'; \ell])} \theta_u \sum_{a \in E_u} P_{a, \alpha[u]}^{(u)} \int_{s_{\ell}} f(S_{u}^{a}(\alpha)[\ell': \ell], s_{\ell})
\]
\[
+ \sum_{u \in B(\alpha[\ell'; \ell])} \rho_u \int_{s_{u_1}} f(\alpha[\ell': u_1], s_{u_1}) \int_{s_{\ell}} f(\alpha[u_2, \ell], s_{\ell})
\]
where the first equality is by definition (S.4), and the second equality obtained by exchanging the integrals and making the substitution \( t_{\ell} \rightarrow t_{\ell} - t_p \). Thus, \( \int_{s_{\ell}} f(\alpha[\ell': \ell], s_{\ell}) \) satisfies the recursion for \( \hat{\pi}_{PS,1} \) (Paul and Song, 2010, Equation (12)) and we conclude that \( \int_{s_{\ell}} f(\alpha[\ell': \ell], s_{\ell}) = \hat{\pi}_{PS,1}(\alpha[\ell': \ell]) \).
Thus,
\[
\hat{\pi}_{\text{SMC}}(\alpha[\ell': \ell]) = \int_{s_{\ell}} f_{\text{SMC}}(\alpha[\ell': \ell], s_{\ell}) = \int_{s_{\ell}} f(\alpha[\ell': \ell], s_{\ell}) = \hat{\pi}_{PS,1}(\alpha[\ell': \ell]),
\]
thereby establishing the desired identity. \( \square \)

**LITERATURE CITED**
