File S1
Supporting Information

PROOF OF EQUIVALENCE OF $\hat{\pi}_{\text{SMC}}$ AND $\hat{\pi}_{\text{PS},1}$

We now want to give a more detailed proof of Proposition 1 from the main text. With the notation as in the paper we have:

Proposition 1. *For an arbitrary single haplotype $\alpha \in \mathcal{H}$ and haplotype configuration* $\mathbf{n}$, $\hat{\pi}_{\text{SMC}}(\alpha|\mathbf{n}) = \hat{\pi}_{\text{PS},1}(\alpha|\mathbf{n})$.

Recall that the initial (stationary) density was given by

$$\zeta^{(\mathbf{n})}(S_\ell = (t_\ell, h_\ell)) = \frac{n_{h_\ell}}{2} e^{-\frac{n}{2}t_\ell}, \tag{S.1}$$

the transtition density by

$$\phi^{(\mathbf{n})}_{\rho_b}(s_\ell \mid s_{\ell-1}) = e^{-\frac{\rho_b}{2}t_{\ell-1}}\delta_{s_{\ell-1},s_\ell} + \frac{n_{\alpha_\ell}}{n}\int_{t_b=0}^{t_{\ell-1}\wedge t_\ell} \frac{\rho_b}{2}e^{-\frac{\rho_b}{2}t_b}\frac{n}{2}e^{-\frac{n}{2}(t_\ell - t_b)}, \tag{S.2}$$

and the emission probability by

$$\xi^{(\mathbf{n})}_{\theta_\ell}(\alpha[\ell] \mid s_\ell) = \left[e^{\frac{\theta_\ell}{2}t_\ell\cdot(P^{(\ell)}-I)}\right]_{h_\ell[\ell],\alpha[\ell]}. \tag{S.3}$$

Since the configuration $\mathbf{n}$ is fixed, we will drop the superscript $(\mathbf{n})$ in the sequel. As in the main text we will also omit the recombination and mutation rate when unambiguous. Further, we will omit the $d\cdot$ whenever we write down integrals. If not specified differently, equation-references refer to equations from the main paper.

Proof of Proposition 1. We start by showing inductively that the joint density $f_{\text{SMC}}(\alpha[\ell' : \ell], (t_\ell, h_\ell))$ of observing the partial haplotype $\alpha[\ell' : \ell]$ and being in the hidden state (t_ℓ, h_ℓ) (basically) introduced in equations (4)-(6) satisfies a genealogical recursion f, defined as follows [c.f., Griffiths and Tavaré (1994)]:

$$\begin{aligned} f(\alpha[\ell' : \ell], (t_\ell, h_\ell)) = \int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u \theta_u + \sum_u \rho_u}{2}t_p}\Bigg[&\frac{n_{h_\ell}\delta_{\alpha[\ell':\ell],h_\ell[\ell':\ell]}}{2}\delta_{t_p,t_\ell} \\ &+ \sum_{u\in L(\ell':\ell)}\frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell' : \ell], (t_\ell - t_p, h_\ell)) \\ &+ \sum_{u\in B(\ell':\ell)}\frac{\rho_u}{2}\left(\int_{s_{u_1}} f(\alpha[\ell' : u_1], s_{u_1})\right) f(\alpha[u_2, \ell], (t_\ell - t_p, h_\ell))\Bigg], \end{aligned} \tag{S.4}$$

where the sum $\sum_u$ is over $L(\ell' : \ell)$, all loci between (and including) ℓ' and ℓ, and $\sum_u$ is over $B(\ell' : \ell)$, the breakpoints between ℓ' and ℓ. Here $\mathcal{S}^a_u(\alpha)$ denotes the haplotype obtained by substituting the allele a at locus u of α, and $u = (u_1, u_2)$. For $\ell' = \ell$ (so that $\ell - \ell' = 0$),

$$f(\alpha[\ell], s_\ell) = \int_{t_p=0}^{t_\ell} e^{-\frac{n+\theta_\ell}{2}t_p}\left[\frac{n_{h_\ell}\delta_{\alpha[\ell],h_\ell[\ell]}}{2}\delta_{t_p,t_\ell} + \frac{\theta}{2}\sum_{a\in E_\ell} P^{(\ell)}_{a,\alpha[\ell]} f(a, (t_\ell - t_p, h_\ell))\right].$$

Substituting $f = f_{\text{SMC}}$ on the right-hand side,

$$\begin{aligned}
&\int_{t_p=0}^{t_\ell} e^{-\frac{n+\theta_\ell}{2}t_p}\left[\frac{n_{h_\ell}\delta_{\alpha[\ell],h_\ell[\ell]}}{2}\delta_{t_p,t_\ell} + \frac{\theta_\ell}{2}\sum_{a\in E_\ell} P^{(\ell)}_{a,\alpha[\ell]} f_{\text{SMC}}(a,(t_\ell - t_p, h_\ell))\right] \\
&\quad= e^{-\frac{n+\theta_\ell}{2}t_\ell}\frac{n_{h_\ell}\delta_{\alpha[\ell],h_\ell[\ell]}}{2} + \int_{t_p=0}^{t_\ell} e^{-\frac{n+\theta}{2}t_p}\frac{\theta_\ell}{2}\sum_{a\in E_\ell} P^{(\ell)}_{a,\alpha[\ell]} p(a \mid (t_\ell - t_p, h_\ell))\cdot q_1(t_u - t_p, h_u) \\
&\quad= \frac{n_{h_\ell}}{2}e^{-\frac{n+\theta_\ell}{2}t_\ell}\left(\delta_{\alpha[\ell],h_\ell[\ell]} + \sum_{m=0}^{\infty}\left(\sum_{a\in E_\ell} P^{(\ell)}_{a,h[\ell]}\left[(P^{(\ell)})^m\right]_{h_\ell[\ell],a}\right)\int_{t_p=0}^{t_\ell}\frac{\theta_\ell}{2}\frac{\left(\frac{\theta_\ell}{2}(t_\ell - t_p)\right)^m}{m!}\right) \\
&\quad= \frac{n_{h_\ell}}{2}e^{-\frac{n+\theta_\ell}{2}t_\ell}\left(\delta_{\alpha[\ell],h_\ell[\ell]} + \sum_{m=0}^{\infty}\left[(P^{(\ell)})^{m+1}\right]_{h_\ell[\ell],\alpha[\ell]}\frac{\left(\frac{\theta_\ell}{2}(t_\ell)\right)^{m+1}}{(m+1)!}\right) \\
&\quad= \frac{n_{h_\ell}}{2}e^{-\frac{n}{2}t_\ell}\cdot\left[e^{\frac{\theta_\ell}{2}t_\ell\cdot(P^{(\ell)}-I)}\right]_{h_\ell[\ell],\alpha[\ell]},
\end{aligned}$$

with the final result equal to $\xi(\alpha[\ell] \mid s_\ell)\zeta(s_\ell) = f_{\text{SMC}}(\alpha[\ell], s_\ell)$. Now, inductively assuming that $f_{\text{SMC}}(\alpha[\ell' : \ell], s_\ell) = f(\alpha[\ell' : \ell], s_\ell)$ for $0 \le \ell - \ell' < m$ and all values of s_ℓ, let $\ell' < \ell$ such that $\ell - \ell' = m$. Substituting $f = f_{\text{SMC}}$ on the right-hand side of (S.4), we obtain

$$\begin{aligned}
&\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u\theta_u+\sum_u\rho_u}{2}t_p}\left[\frac{n_{h_\ell}\delta_{\alpha[\ell':\ell],h_\ell[\ell':\ell]}}{2}\delta_{t_p,t_\ell}\right. \\
&\quad+ \sum_{u\in L(\ell':\ell)}\frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f_{\text{SMC}}(\mathcal{S}^a_u(\alpha)[\ell' : \ell],(t_\ell - t_p, h_\ell)) \\
&\quad+ \left.\sum_{u\in B(\ell':\ell)}\frac{\rho_u}{2}\left(\int_{s_{u_1}} f_{\text{SMC}}(\alpha[\ell' : u_1], s_{u_1})\right) f_{\text{SMC}}(\alpha[u_2,\ell],(t_\ell - t_p, h_\ell))\right]. \qquad \text{(S.5)}
\end{aligned}$$

We consider this expression one term at a time. Beginning with the first term:

$$\begin{aligned}
&\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u\theta_u+\sum_u\rho_u}{2}t_p}\frac{n_{h_\ell}\delta_{\alpha[\ell':\ell],h_\ell[\ell':\ell]}}{2}\delta_{t_p,t_\ell} \\
&\quad= \int_{s_{\ell-1}}\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u\theta_u+\sum'_u\rho_u}{2}t_p}\frac{n_{h_{\ell-1}}\delta_{\alpha[\ell':\ell-1],h_{\ell-1}[\ell':\ell-1]}}{2}\delta_{t_p,t_{\ell-1}}\left[e^{-\frac{\theta_\ell+\rho_b}{2}t_p}\delta_{\alpha[\ell],h_\ell[\ell]}\delta_{s_{\ell-1},s_\ell}\right], \qquad \text{(S.6)}
\end{aligned}$$

where $\sum'_u$ is over $L(\ell' : \ell-1)$ and $\sum'_u$ is over $B(\ell' : \ell-1)$. Moving on to the second term, expand using the definition (5) of f_{SMC}, and then use the inductive hypothesis to replace the resulting f_{SMC} terms with the corresponding f terms:

$$\begin{aligned}
&\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u\theta_u+\sum_u\rho_u}{2}t_p}\sum_{u\in L(\ell':\ell)}\frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f_{\text{SMC}}(\mathcal{S}^a_u(\alpha)[\ell' : \ell],(t_\ell - t_p, h_\ell)) \\
&\quad= \int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u\theta_u+\sum_u\rho_u}{2}t_p}\sum_{u\in L(\ell':\ell-1)}\frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} \\
&\qquad\times\xi(\alpha[\ell] \mid (t_\ell - t_p, h_\ell))\int_{s_{\ell-1}}\phi((t_\ell - t_p, h_\ell) \mid s_{\ell-1}) f(\mathcal{S}^a_u(\alpha)[\ell' : \ell-1], s_{\ell-1}) \\
&\quad+ \int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u\theta_u+\sum_u\rho_u}{2}t_p}\frac{\theta_\ell}{2}\sum_{a\in E_u} P_{a,\alpha[\ell]} \\
&\qquad\times\xi(a \mid (t_\ell - t_p, h_\ell))\int_{s_{\ell-1}}\phi((t_\ell - t_p, h_\ell) \mid s_{\ell-1}) f(\alpha[\ell' : \ell-1], s_{\ell-1}).
\end{aligned}$$

Concentrating on the first sub-term, making the substitution $t_{\ell-1} \to t_{\ell-1} + t_p$, and changing the order of integration, we obtain

$$\int_{s_{\ell-1}} \int_{t_p=0}^{t_\ell \wedge t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u + \sum'_u \rho_u}{2} t_p} \sum_{u \in L(\ell':\ell-1)} \frac{\theta_u}{2} \sum_{a \in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell' : \ell-1], (t_{\ell-1} - t_p, h_{\ell-1}))$$
$$\times \left[e^{-\frac{\theta_\ell}{2} t_p} p(\alpha[\ell] \mid (t_\ell - t_p, h_\ell)) \cdot e^{-\frac{\rho_b}{2} t_p} q((t_\ell - t_p, h_\ell) \mid (t_{\ell-1} - t_p, h_{\ell-1})) \right]. \quad \text{(S.7)}$$

Now concentrating on the second sub-term and expanding using definition (S.4) of f:

$$\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u \theta_u + \sum_u \rho_u}{2} t_p} \frac{\theta_\ell}{2} \sum_{a \in E_u} P_{a,\alpha[\ell]} \xi(a \mid (t_\ell - t_p, h_\ell)) \int_{s_{\ell-1}} \phi((t_\ell - t_p, h_\ell) \mid s_{\ell-1})$$
$$\times \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u + \sum'_u \rho_u}{2} t_q} \left[\frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_q, t_{\ell-1}} \right.$$
$$+ \sum_{u \in L(\ell':\ell-1)} \frac{\theta_u}{2} \sum_{a \in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell' : \ell-1], (t_{\ell-1} - t_q, h_{\ell-1}))$$
$$\left. + \sum_{u \in B(\ell':\ell-1)} \frac{\rho_u}{2} \left(\int_{s_{u_1}} f(\alpha[\ell' : u_1], s_{u_1}) \right) f(\alpha[u_2, \ell-1], (t_{\ell-1} - t_q, h_{\ell-1})) \right]$$
$$= \int_{s_{\ell-1}} \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u + \sum'_u \rho_u}{2} t_q} \left[\frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_q, t_{\ell-1}} \right.$$
$$+ \sum_{u \in L(\ell':\ell-1)} \frac{\theta_u}{2} \sum_{a \in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell' : \ell-1], (t_{\ell-1} - t_q, h_{\ell-1}))$$
$$\left. + \sum_{u \in B(\ell':\ell-1)} \frac{\rho_u}{2} \left(\int_{s_{u_1}} f(\alpha[\ell' : u_1], s_{u_1}) \right) f(\alpha[u_2, \ell-1], (t_{\ell-1} - t_q, h_{\ell-1})) \right]$$
$$\times \left[\int_{t_p=0}^{t_q \wedge t_\ell} e^{-\frac{\theta_\ell}{2} t_p} \frac{\theta_\ell}{2} \sum_{a \in E_u} P_{a,\alpha[\ell]} \xi(a \mid (t_\ell - t_p, h_\ell)) \cdot e^{-\frac{\rho_b}{2} t_p} \phi((t_\ell - t_p, h_\ell) \mid (t_{\ell-1} - t_p, h_{\ell-1})) \right], \quad \text{(S.8)}$$

with the equality obtained by making the substitutions $t_{\ell-1} \to t_{\ell-1} + t_p$ and $t_q \to t_q + t_p$ and then changing the order of integration. Finally, moving onto the third term, expand using the definition (5) of f_{SMC}, and then use the inductive hypothesis to replace the resulting f_{SMC} terms with the corresponding f terms:

$$\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u \theta_u + \sum_u \rho_u}{2} t_p} \sum_{u \in B(\ell':\ell)} \frac{\rho_u}{2} \left(\int_{s_{u_l}} f_{\text{SMC}}(\alpha[\ell' : u_l], s_{u_l}) \right) f_{\text{SMC}}(\alpha[u_r, \ell], (t_\ell - t_p, h_\ell))$$
$$= \int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u \theta_u + \sum_u \rho_u}{2} t_p} \sum_{u \in B(\ell':\ell-1)} \frac{\rho_u}{2} \left(\int_{s_{u_l}} f(\alpha[\ell' : u_l], s_{u_l}) \right)$$
$$\times \xi(\alpha[\ell] \mid (t_\ell - t_p, h_\ell)) \int_{s_{\ell-1}} \phi((t_\ell - t_p, h_\ell) \mid s_{\ell-1}) f(\alpha[u_r : \ell-1], s_{\ell-1})$$
$$+ \int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u \theta_u + \sum_u \rho_u}{2} t_p} \frac{\rho_b}{2} \left(\int_{s_{\ell-1}} f(\alpha[\ell' : \ell-1], s_{\ell-1}) \right) \cdot f(\alpha[\ell], (t_\ell - t_p, h_\ell)).$$

Concentrating on the first sub-term, making the substitution $t_{\ell-1} \to t_{\ell-1} + t_p$, and changing the order of integration, we obtain:

$$\int_{s_{\ell-1}} \int_{t_p=0}^{t_\ell \wedge t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u + \sum'_u \rho_u}{2} t_p} \sum_{u \in B(\ell':\ell-1)} \frac{\rho_u}{2} \left(\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1}) \right) f(\alpha[u_2:\ell-1], (t_{\ell-1} - t_p, h_{\ell-1}))$$
$$\times \left[e^{-\frac{\theta_\ell}{2} t_p} \xi(\alpha[\ell] \mid (t_\ell - t_p, h_\ell)) \cdot e^{-\frac{\rho_b}{2} t_p} \phi((t_\ell - t_p, h_\ell) \mid (t_{\ell-1} - t_p, h_{\ell-1})) \right]. \tag{S.9}$$

Now concentrating on the second sub-term and expanding using definition (S.4) of f:

$$\begin{aligned}
&\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u \theta_u + \sum_u \rho_u}{2} t_p} \frac{\rho_b}{2} f(\alpha[\ell], (t_\ell - t_p, h_\ell)) \\
&\quad \times \int_{s_{\ell-1}} \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u + \sum'_u \rho_u}{2} t_q} \Bigg[\frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_q, t_{\ell-1}} \\
&\qquad + \sum_{u \in L(\ell':\ell-1)} \frac{\theta_u}{2} \sum_{a \in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1], (t_{\ell-1} - t_q, h_{\ell-1})) \\
&\qquad + \sum_{u \in B(\ell':\ell-1)} \frac{\rho_u}{2} \left(\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1}) \right) f(\alpha[u_2, \ell-1], (t_{\ell-1} - t_q, h_{\ell-1})) \Bigg] \\
&= \int_{s_{\ell-1}} \int_{t_q=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u + \sum'_u \rho_u}{2} t_q} \Bigg[\frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_q, t_{\ell-1}} \\
&\qquad + \sum_{u \in L(\ell':\ell-1)} \frac{\theta_u}{2} \sum_{a \in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1], (t_{\ell-1} - t_q, h_{\ell-1})) \\
&\qquad + \sum_{u \in B(\ell':\ell-1)} \frac{\rho_u}{2} \left(\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1}) \right) f(\alpha[u_2, \ell-1], (t_{\ell-1} - t_q, h_{\ell-1})) \Bigg] \\
&\quad \times \left[\int_{t_p=0}^{t_q \wedge t_\ell} e^{-\frac{\theta_\ell}{2} t_p} \xi(\alpha[\ell] \mid (t_\ell - t_p, h_\ell)) \cdot e^{-\frac{\rho_b}{2} t_p} \frac{\rho_b}{2} \frac{n_{h_\ell}}{2} e^{-\frac{n}{2}(t_\ell - t_p)} \right],
\end{aligned} \tag{S.10}$$

with the equality obtained by using the (one-locus) definition (6) for $f_{\text{SMC}}(\alpha[\ell], (t_\ell - t_p, h_\ell))$, making the substitutions $t_{\ell-1} \to t_{\ell-1} + t_p$ and $t_q \to t_q + t_p$, and changing the order of integration.

Having appropriately expanded each term of our key expression (S.5), we aggregate common terms across the resulting sub-expressions. Collecting the $n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}$ terms from (S.6),(S.8), and (S.10),

$$
\begin{aligned}
&\int_{s_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p} \frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_p, t_{\ell-1}} \\
&\quad \times \Bigg[e^{-\frac{\theta_\ell+\rho_b}{2} t_p} \delta_{\alpha[\ell], h_\ell[\ell]} \delta_{s_{\ell-1}, s_\ell} \\
&\qquad + \int_{t_q=0}^{t_p \wedge t_\ell} e^{-\frac{\theta_\ell}{2} t_q} \frac{\theta_\ell}{2} \sum_{a \in E_u} P_{a,\alpha[\ell]} \xi(a \mid (t_\ell - t_q, h_\ell)) \cdot e^{-\frac{\rho_b}{2} t_q} \phi((t_\ell - t_q, h_\ell) \mid (t_{\ell-1} - t_q, h_{\ell-1})) \\
&\qquad + \int_{t_q=0}^{t_p \wedge t_\ell} e^{-\frac{\theta_\ell}{2} t_q} \xi(\alpha[\ell] \mid (t_\ell - t_q, h_\ell)) \cdot e^{-\frac{\rho_b}{2} t_q} \frac{\rho_b}{2} \frac{n_{h_\ell}}{2} e^{-\frac{n}{2}(t_\ell - t_q)} \Bigg] \\
&= \int_{s_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p} \frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_p, t_{\ell-1}} \\
&\quad \times \Bigg[e^{-\frac{\rho_b}{2} t_{\ell-1}} \delta_{s_{\ell-1}, s_\ell} \cdot \left(e^{-\frac{\theta_\ell}{2} t_\ell} \delta_{\alpha[\ell], h_\ell[\ell]} \right) \\
&\qquad + e^{-\frac{\rho_b}{2} t_{\ell-1}} \delta_{s_{\ell-1}, s_\ell} \left(\int_{t_z=0}^{t_\ell} e^{-\frac{\theta_\ell}{2} t_z} \frac{\theta_\ell}{2} \sum_{a \in E_u} P_{a,\alpha[\ell]} \xi(a \mid (t_\ell - t_z, h_\ell)) \right) \\
&\qquad + \int_{t_q=0}^{t_{\ell-1} \wedge t_\ell} \frac{\rho_b}{2} e^{-\frac{\rho_b}{2} t_q} \frac{n_{h_\ell}}{2} e^{-\frac{n}{2}(t_\ell - t_q)} \left(\int_{t_z=0}^{t_q} e^{-\frac{\theta_\ell}{2} t_z} \frac{\theta_\ell}{2} \sum_{a \in E_u} P_{a,\alpha[\ell]} \xi(allele \mid (t_\ell - t_z, h_\ell)) \right) \\
&\qquad + \int_{t_q=0}^{t_{\ell-1} \wedge t_\ell} \frac{\rho_b}{2} e^{-\frac{\rho_b}{2} t_q} \frac{n_{h_\ell}}{2} e^{-\frac{n}{2}(t_\ell - t_q)} \left(e^{-\frac{\theta_\ell}{2} t_q} \xi(\alpha[\ell] \mid (t_\ell - t_q, h_\ell)) \right) \Bigg] \\
&= \int_{s_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p} \frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_p, t_{\ell-1}} \\
&\quad \times \xi(\alpha[\ell] \mid s_\ell) \left[e^{-\frac{\rho_b}{2} t_{\ell-1}} \delta_{s_{\ell-1}, s_\ell} + \int_{t_q=0}^{t_{\ell-1} \wedge t_\ell} \frac{\rho_b}{2} e^{-\frac{\rho_b}{2} t_q} \frac{n_{h_\ell}}{2} e^{-\frac{n}{2}(t_\ell - t_q)} \right] \\
&= \int_{s_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p} \frac{n_{h_{\ell-1}} \delta_{\alpha[\ell':\ell-1], h_{\ell-1}[\ell':\ell-1]}}{2} \delta_{t_p, t_{\ell-1}} \times \Big[\xi(\alpha[\ell] \mid s_\ell) \phi(s_\ell \mid s_{\ell-1}) \Big],
\end{aligned} \tag{S.11}
$$

where the first equality is obtained by making use of the $\delta_{t_p,t_{\ell-1}}$ and $\delta_{s_{\ell-1},s_\ell}$ expressions and expanding the q term using equation (S.2) and exchanging integrals, the second equality is obtained by combining the first/second and third/fourth term along with the definition (S.3) of p, and final equality by again making use of the equation (S.2).

Similarly, collecting the $f(\mathcal{S}_u^a(\alpha)[\ell' : \ell-1], (t_{\ell-1}-t_q, h_{\ell-1}))$ terms from the resulting sub-expressions

(S.7),(S.8), and (S.10),

$$\begin{aligned}
&\int_{s_{\ell-1}}\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_u' \theta_u+\sum_u' \rho_u}{2}t_p} \sum_{u\in L(\ell':\ell-1)} \frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1],(t_{\ell-1}-t_p,h_{\ell-1}))\\
&\times\Bigg[\mathbb{I}_{(t_p\le t_\ell)} e^{-\frac{\theta_\ell}{2}t_p}\xi(\alpha[\ell]\mid(t_\ell-t_p,h_\ell))\cdot e^{-\frac{\rho_b}{2}t_p}\phi((t_\ell-t_p,h_\ell)\mid(t_{\ell-1}-t_p,h_{\ell-1}))\\
&\quad+\int_{t_q=0}^{t_p\wedge t_\ell} e^{-\frac{\theta_\ell}{2}t_q}\frac{\theta_\ell}{2}\sum_{a\in E_u} P_{a,\alpha[\ell]}\xi(a\mid(t_\ell-t_q,h_\ell))\cdot e^{-\frac{\rho_b}{2}t_q}\phi((t_\ell-t_q,h_\ell)\mid(t_{\ell-1}-t_q,h_{\ell-1}))\\
&\quad+\int_{t_q=0}^{t_p\wedge t_\ell} e^{-\frac{\theta_\ell}{2}t_q}\xi(\alpha[\ell]\mid(t_\ell-t_q,h_\ell))\cdot e^{-\frac{\rho_b}{2}t_q}\frac{\rho_b}{2}\frac{n_{h_\ell}}{2}e^{-\frac{n}{2}(t_\ell-t_q)}\Bigg]\\
&=\int_{s_{\ell-1}}\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_u' \theta_u+\sum_u' \rho_u}{2}t_p} \sum_{u\in L(\ell':\ell-1)} \frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1],(t_{\ell-1}-t_p,h_{\ell-1}))\\
&\times\Bigg[\mathbb{I}_{(t_p\le t_\ell)} e^{-\frac{\rho_b}{2}t_p}\phi((t_\ell-t_p,h_\ell)\mid(t_{\ell-1}-t_p,h_{\ell-1}))\Big(e^{-\frac{\theta_\ell}{2}t_p}\xi(\alpha[\ell]\mid(t_\ell-t_p,h_\ell))\Big)\\
&\quad+\mathbb{I}_{(t_p\le t_\ell)} e^{-\frac{\rho_b}{2}t_p}\phi((t_\ell-t_p,h_\ell)\mid(t_{\ell-1}-t_p,h_{\ell-1}))\Big(\int_{t_z=0}^{t_p} e^{-\frac{\theta_\ell}{2}t_z}\frac{\theta_\ell}{2}\sum_{a\in E_u}P_{a,\alpha[\ell]}\xi(a\mid(t_\ell-t_z,h_\ell))\Big)\\
&\quad+\int_{t_q=0}^{t_p\wedge t_\ell}\frac{\rho_b}{2}e^{-\frac{\rho_b}{2}t_q}\frac{n_{h_\ell}}{2}e^{-\frac{n}{2}(t_\ell-t_q)}\Big(\int_{t_z=0}^{t_q} e^{-\frac{\theta_\ell}{2}t_z}\frac{\theta_\ell}{2}\sum_{a\in E_u}P_{a,\alpha[\ell]}\xi(a\mid(t_\ell-t_z,h_\ell))\Big)\\
&\quad+\int_{t_q=0}^{t_p\wedge t_\ell}\frac{\rho_b}{2}e^{-\frac{\rho_b}{2}t_q}\frac{n_{h_\ell}}{2}e^{-\frac{n}{2}(t_\ell-t_q)}\Big(e^{-\frac{\theta_\ell}{2}t_q}\xi(\alpha[\ell]\mid(t_\ell-t_q,h_\ell))\Big)\Bigg]\\
&=\int_{s_{\ell-1}}\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_u' \theta_u+\sum_u' \rho_u}{2}t_p} \sum_{u\in L(\ell':\ell-1)} \frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1],(t_{\ell-1}-t_p,h_{\ell-1}))\\
&\times\xi(\alpha[\ell]\mid s_\ell)\Bigg[\mathbb{I}_{(t_p\le t_\ell)} e^{-\frac{\rho_b}{2}t_p}\phi((t_\ell-t_p,h_\ell)\mid(t_{\ell-1}-t_p,h_{\ell-1}))+\int_{t_q=0}^{t_p\wedge t_\ell}\frac{\rho_b}{2}e^{-\frac{\rho_b}{2}t_q}\frac{n_{h_\ell}}{2}e^{-\frac{n}{2}(t_\ell-t_q)}\Bigg]\\
&=\int_{s_{\ell-1}}\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum_u' \theta_u+\sum_u' \rho_u}{2}t_p} \sum_{u\in L(\ell':\ell-1)} \frac{\theta_u}{2}\sum_{a\in E_u} P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1],(t_{\ell-1}-t_p,h_{\ell-1}))\\
&\times\Bigg[\xi(\alpha[\ell]\mid s_\ell)\phi(s_\ell\mid s_{\ell-1})\Bigg],
\end{aligned} \tag{S.12}$$

where the first equality is obtained by expanding the ϕ term[1] in the second term using equation (S.2), the second equality is obtained by combining the first/second and third/fourth term along with the definition (S.3) of ξ, and final equality by again making use of the equation (S.2) and considering separately the case when $t_p \le t_\ell$ and $t_p > t_\ell$.

The situation is identical when collecting terms with $f(\alpha[u_2,\ell-1],(t_{\ell-1}-t_q,h_{\ell-1}))$ from (S.9),

[1] We use the following expansion for ϕ, which can be verified in the present context, namely that $t_q \le t_p \le t_{\ell-1}$ and $t_q \le t_\ell$:

$$\begin{aligned}
\phi((t_\ell-t_q,h_\ell)\mid(t_{\ell-1}-t_q,h_{\ell-1})) &= \mathbb{I}_{(t_p\le t_\ell)}e^{-\frac{\rho_b}{2}(t_p-t_q)}\cdot\phi((t_\ell-t_p,h_\ell)\mid(t_{\ell-1}-t_p,h_{\ell-1}))\\
&\quad+\int_{t_z=0}^{(t_p\wedge t_\ell)-t_q}\frac{\rho_b}{2}e^{-\frac{\rho_b}{2}t_z}\frac{n_{h_\ell}}{2}e^{-\frac{n}{2}(t_\ell-t_q-t_z)}
\end{aligned}$$

(S.8), and (S.10):

$$\begin{aligned}
&\int_{s_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p} \sum_{u\in B(\ell':\ell-1)} \frac{\rho_u}{2}\Big(\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1})\Big) f(\alpha[u_2:\ell-1], (t_{\ell-1}-t_p, h_{\ell-1}))\\
&\quad\times\Big[\mathbb{I}_{(t_p\leq t_\ell)} e^{-\frac{\theta_\ell}{2}t_p}\xi(\alpha[\ell] \mid (t_\ell - t_p, h_\ell))\cdot e^{-\frac{\rho_b}{2}t_p}\phi((t_\ell-t_p,h_\ell)\mid (t_{\ell-1}-t_p, h_{\ell-1}))\\
&\qquad + \int_{t_q=0}^{t_p\wedge t_\ell} e^{-\frac{\theta_\ell}{2}t_q}\frac{\theta_\ell}{2}\sum_{a\in E_u} P_{a,\alpha[\ell]}\xi(a\mid (t_\ell-t_q,h_\ell))\cdot e^{-\frac{\rho_b}{2}t_q}\phi((t_\ell-t_q,h_\ell)\mid(t_{\ell-1}-t_q,h_{\ell-1}))\\
&\qquad + \int_{t_q=0}^{t_p\wedge t_\ell} e^{-\frac{\theta_\ell}{2}t_q}\xi(\alpha[\ell]\mid (t_\ell-t_q,h_\ell))\cdot e^{-\frac{\rho_b}{2}t_q}\frac{\rho_b}{2}\frac{n_{h_\ell}}{2}e^{-\frac{n}{2}(t_\ell-t_q)}\Big]\\
&= \int_{s_{\ell-1}} \int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p} \sum_{u\in B(\ell':\ell-1)} \frac{\rho_u}{2}\Big(\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1})\Big) f(\alpha[u_2:\ell-1], (t_{\ell-1}-t_p, h_{\ell-1}))\\
&\quad\times\Big[\xi(\alpha[\ell]\mid s_\ell)\phi(s_\ell\mid s_{\ell-1})\Big]. \qquad\qquad (S.13)
\end{aligned}$$

Thus, combining equations (S.11),(S.12), and (S.13), we may re-write (S.5):

$$\begin{aligned}
&\xi(\alpha[\ell]\mid s_\ell)\int_{s_{\ell-1}}\phi(s_\ell\mid s_{\ell-1})\cdot\int_{t_p=0}^{t_{\ell-1}} e^{-\frac{n+\sum'_u \theta_u+\sum'_u \rho_u}{2} t_p}\Big[\frac{n_{h_{\ell-1}}\delta_{\alpha[\ell':\ell-1],h_{\ell-1}[\ell':\ell-1]}}{2}\delta_{t_p,t_{\ell-1}}\\
&\qquad + \sum_{u\in L(\ell':\ell-1)}\frac{\theta_u}{2}\sum_{a\in E_u}P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell-1], (t_{\ell-1}-t_p,h_{\ell-1}))\\
&\qquad + \sum_{u\in B(\ell':\ell-1)}\frac{\rho_u}{2}\Big(\int_{s_{u_1}} f(\alpha[\ell':u_1],s_{u_1})\Big) f(\alpha[u_2:\ell-1],(t_{\ell-1}-t_p,h_{\ell-1}))\Big]\\
&= \xi(\alpha[\ell]\mid s_\ell)\int_{s_{\ell-1}}\phi(s_\ell\mid s_{\ell-1}) f(\alpha[\ell':\ell-1], s_{\ell-1})\\
&= f_{\text{SMC}}(\alpha[\ell':\ell], s_\ell),
\end{aligned}$$

where the first equality is obtained by definition (S.4) for f, and the second equality by using the inductive hypothesis and the definition (5). Therefore, f_{SMC} satisfies the recursion for f, and we conclude that $f_{\text{SMC}} = f$. Moreover,

$$\begin{aligned}\int_{s_\ell} f(\alpha[\ell':\ell], s_\ell) &= \int_{s_\ell}\int_{t_p=0}^{t_\ell} e^{-\frac{n+\sum_u\theta_u+\sum_u\rho_u}{2}t_p}\Bigg[\frac{n_{h_\ell}\delta_{\alpha[\ell':\ell],h_\ell[\ell':\ell]}}{2}\delta_{t_p,t_\ell} \\ &\quad + \sum_{u\in L(\ell':\ell)}\frac{\theta_u}{2}\sum_{a\in E_u}P^{(u)}_{a,\alpha[u]} f(\mathcal{S}^a_u(\alpha)[\ell':\ell],(t_\ell - t_p, h_\ell)) \\ &\quad + \sum_{u\in B(\ell':\ell)}\frac{\rho_u}{2}\left(\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1})\right) f(\alpha[u_2,\ell],(t_\ell - t_p, h_\ell))\Bigg] \\ &= \frac{1}{n+\sum_{u\in L(\alpha[\ell':\ell])}\theta_u + \sum_{u\in B(\alpha[\ell':\ell])}\rho_u}\Bigg[\sum_{\substack{\alpha'\in\mathcal{H}:\\ \alpha'[\ell':\ell]=\alpha[\ell':\ell]}} n_{\alpha'} \\ &\quad + \sum_{u\in L(\alpha[\ell':\ell])}\theta_u\sum_{a\in E_u}P^{(u)}_{a,\alpha[u]}\int_{s_\ell} f(\mathcal{S}^a_u(\alpha)[\ell':\ell], s_\ell) \\ &\quad + \sum_{u\in B(\alpha[\ell':\ell])}\rho_u\int_{s_{u_1}} f(\alpha[\ell':u_1], s_{u_1})\int_{s_\ell} f(\alpha[u_2,\ell], s_\ell)\Bigg],\end{aligned}$$

where the first equality is by definition (S.4), and the second equality obtained by exchanging the integrals and making the substitution $t_\ell \to t_\ell - t_p$. Thus, $\int_{s_\ell} f(\alpha[\ell':\ell], s_\ell)$ satisfies the recursion for $\hat{\pi}_{\text{PS},1}$ (Paul and Song, 2010, Equation (12)) and we conclude that $\int_{s_\ell} f(\alpha[\ell':\ell], s_\ell) = \hat{\pi}_{\text{PS},1}(\alpha[\ell':\ell])$. Thus,

$$\hat{\pi}_{\text{SMC}}(\alpha[\ell':\ell]) = \int_{s_\ell} f_{\text{SMC}}(\alpha[\ell':\ell], s_\ell) = \int_{s_\ell} f(\alpha[\ell':\ell], s_\ell) = \hat{\pi}_{\text{PS},1}(\alpha[\ell':\ell]),$$

thereby establishing the desired identity. □

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