FILE S1

Part I: Some general properties of the transformation $\mathcal{L}$

We can gain some insight into the eigenvalues of $\mathcal{L}$ by the following observations.

Observation I

\begin{align}
(i) \quad \det(\mathcal{L}_i^\text{ex}) &= \frac{(1 - 2\mu_m)(1 - r)w_1^i w_3^i}{(w^i)^2} \quad \text{(A1)} \\
(ii) \quad \det(\mathcal{L}_\text{ex}) &= \frac{(1 - 2\mu_m)^k(1 - r)^k \prod_{i=1}^{k} w_1^i w_3^i}{w^2} \quad \text{(A2)}
\end{align}

The computation of $\det(\mathcal{L}_i^\text{ex})$ is straightforward and follows from the fact that $\mathcal{L}_\text{ex} = \mathcal{L}_k \cdot \mathcal{L}_{k-1} \cdot \cdots \cdot \mathcal{L}_1$.

Clearly when $\mu_m = \frac{1}{2}$, one of the eigenvalues of $\mathcal{L}_\text{ex}$ is zero. Also, if $0 \leq r < 1$ and $0 \leq \mu_m < \frac{1}{2}$, the two eigenvalues are positive for all $k$, whereas when $\frac{1}{2} < \mu_m \leq 1$ the two eigenvalues are positive for $k$ even and when $k$ is odd, one eigenvalue is positive, and the other negative.

Observation II.

When $r = 1$, the eigenvalues of each of the matrices $\mathcal{L}_i^\text{ex}$ and $\mathcal{L}_\text{ex}$ are zero and one. In fact, when $r = 1$ the matrix $\mathcal{L}_i^\text{ex}$ has the form

$$\mathcal{L}_i^\text{ex} = \frac{1}{a_i + b_i} \begin{bmatrix} a_i & a_i \\ b_i & b_i \end{bmatrix},$$

with

$$a_i = (1 - \mu_m)w_1^i y_1^i + \mu_m w_3^i y_3^i$$
$$b_i = \mu_m w_1^i y_1^i + (1 - \mu_m)w_3^i y_3^i$$

$$a_i + b_i = w_1^i y_1^i + w_3^i y_3^i = w^i.$$  \hspace{1cm} \text{(A4)}

Thus the two eigenvalues of $\mathcal{L}_i^\text{ex}$ are zero and one. Moreover it is easily seen that

$$\mathcal{L}_i^\text{ex} \cdot \mathcal{L}_j^\text{ex} = \frac{1}{a_i + b_i} \begin{bmatrix} a_i & a_i \\ b_i & b_i \end{bmatrix} = \mathcal{L}_i^\text{ex}.$$  \hspace{1cm} \text{(A5)}

Hence $\mathcal{L}_\text{ex} = \mathcal{L}_k^\text{ex} \cdot \mathcal{L}_{k-1}^\text{ex} \cdot \cdots \cdot \mathcal{L}_1^\text{ex}$ also has the form (A3), and so its eigenvalues are zero and one when $r = 1$. 

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Part II: Proof of Equation (29)

Let \( \hat{x} = (\hat{x}_1, 0, \hat{x}_3, 0) \) be such that \( \hat{x} = T_1(x^*) \), namely

\[
\begin{align*}
    w\hat{x}_1 &= (1 - \mu_M)w_1x_1^* + \mu_M w_3 x_3^*, \\
    w\hat{x}_3 &= (1 - \mu_M)w_3 x_3^* + \mu_M w_1 x_1^*,
\end{align*}
\]

(A6)

with

\[
    w = w_1 x_1^* + w_3 x_3^*.
\]

(A7)

Also let

\[
    \hat{w} = \hat{w}_1 \hat{x}_1 + \hat{w}_3 \hat{x}_3.
\]

(A8)

Then the external stability of \( x^* \) is determined by the largest positive eigenvalue of the matrix product \( \mathcal{L}_{\text{ex}} = \hat{\mathcal{L}} \cdot \mathcal{L} \) where

\[
    w\mathcal{L} = \begin{bmatrix}
        (1 - \mu_m)w_1 - r[(1 - \mu_m)w_1 - \mu_m w_3]x_3^* & \mu_m w_3 + r[(1 - \mu_m)w_1 - \mu_m w_3]x_1^* \\
        \mu_m w_1 + r[(1 - \mu_m)w_3 - \mu_m w_1]x_1^* & (1 - \mu_m)w_3 - r[(1 - \mu_m)w_3 - \mu_m w_1]x_1^*
    \end{bmatrix},
\]

and

\[
    \hat{w}\hat{\mathcal{L}} = \begin{bmatrix}
        (1 - \mu_m)\hat{w}_1 - r[(1 - \mu_m)\hat{w}_1 - \mu_m \hat{w}_3]\hat{x}_3 & \mu_m \hat{w}_3 + r[(1 - \mu_m)\hat{w}_1 - \mu_m \hat{w}_3]\hat{x}_1 \\
        \mu_m \hat{w}_1 + r[(1 - \mu_m)\hat{w}_3 - \mu_m \hat{w}_1]\hat{x}_1 & (1 - \mu_m)\hat{w}_3 - r[(1 - \mu_m)\hat{w}_3 - \mu_m \hat{w}_1]\hat{x}_1
    \end{bmatrix}.
\]

(A9)

At equilibrium, we write \( w\hat{w} = w^* \), and following (26) we have

\[
    w^* = \hat{w}_1 w_1 x_1^* + \hat{w}_3 w_3 x_3^* + \mu_M (\hat{w}_3 - \hat{w}_1)(w_1 x_1^* - w_3 x_3^*).
\]

(A11)

We can verify that

\[
    \det(\mathcal{L}_{\text{ex}}) = \frac{(1 - 2\mu_m)^2(1 - r)^2 w_1 \hat{w}_1 w_3 \hat{w}_3}{(w^*)^2}.
\]

(A12)

Since \( \mathcal{L}_{\text{ex}} \) is a positive matrix, it has a positive eigenvalue, and (A12) ensures that \( \det(\mathcal{L}_{\text{ex}}) \) is positive when \( \mu_m \neq \frac{1}{2} \) and \( 0 \leq r < 1 \). If \( \mu_m = \frac{1}{2} \) or \( r = 1 \), \( \mathcal{L}_{\text{ex}} \) has a zero eigenvalue. Thus for all \( \mu_m \neq \frac{1}{2} \), \( \mathcal{L}_{\text{ex}} \) has one positive eigenvalue (the larger in magnitude) and another eigenvalue which is non-negative. The two eigenvalues are the roots of the characteristic polynomial \( C(z) = \det(\mathcal{L}_{\text{ex}} - z\mathcal{I}) \), or equivalently the roots of \( M(z) = \det(w^* \mathcal{L}_{\text{ex}} - w^* z\mathcal{I}) \).

The larger eigenvalue of \( \mathcal{L}_{\text{ex}} \) is less than one if \( M(1) > 0 \) and \( M'(1) > 0 \), and greater than one when \( M(1) < 0 \). \( M(1) \) has the useful representation

\[
    M(1) = \frac{(\mu_m - \mu_M)(1 - r)}{x_1^*} \cdot \Delta,
\]

(A13)
where \( \Delta = \Delta(r, \mu_m) = d_{11}d_{22} - d_{12}d_{21} \) and \( \Delta(r, \mu_m) \) is a bilinear function of \( r \) and \( \mu_m \) in \( 0 \leq r \leq 1, 0 \leq \mu_m \leq 1 \).

We begin by representing \( w\mathbf{L} \) and \( \hat{w}\hat{\mathbf{L}} \) as follows:

\[
\begin{align*}
w\mathbf{L} &= \begin{bmatrix}
(1 - \mu_m)w_1 & \mu_m w_3 \\
\mu_m w_1 & (1 - \mu_m) w_3
\end{bmatrix} - r \begin{bmatrix}
\alpha x_3^* & -\alpha x_1^* \\
-\beta x_3^* & \beta x_1^*
\end{bmatrix} \\
\hat{w}\hat{\mathbf{L}} &= \begin{bmatrix}
(1 - \mu_m)\hat{w}_1 & \mu_m \hat{w}_3 \\
\mu_m \hat{w}_1 & (1 - \mu_m) \hat{w}_3
\end{bmatrix} - r \begin{bmatrix}
\hat{\alpha} \hat{x}_3 & -\hat{\alpha} \hat{x}_1 \\
-\hat{\beta} \hat{x}_3 & \hat{\beta} \hat{x}_1
\end{bmatrix},
\end{align*}
\]

where

\[
\begin{align*}
\alpha &= w_1 - \mu_m(w_1 + w_3) & \beta &= w_3 - \mu_m(w_1 + w_3) \\
\hat{\alpha} &= \hat{w}_1 - \mu_m(\hat{w}_1 + \hat{w}_3) & \hat{\beta} &= \hat{w}_3 - \mu_m(\hat{w}_1 + \hat{w}_3).
\end{align*}
\]

We can now compute \( w^*\mathbf{L}_{\text{ex}} = \hat{w}\hat{\mathbf{L}} \cdot w\mathbf{L} \) (using \( w^* = w\hat{w} \)) as follows:

\[
\begin{align*}
w^*\mathbf{L}_{\text{ex}} &= \begin{bmatrix}
(1 - \mu_m)^2 w_1 \hat{w}_1 + \mu_m^2 w_1 \hat{w}_3 & \mu_m(1 - \mu_m) w_3 (\hat{w}_1 + \hat{w}_3) \\
\mu_m(1 - \mu_m) w_1 (\hat{w}_1 + \hat{w}_3) & (1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1
\end{bmatrix} \\
&\quad - r \begin{bmatrix}
x_3^* [(1 - \mu_m)\hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta] & -x_1^* [(1 - \mu_m)\hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta] \\
-x_3^* [(1 - \mu_m)\hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha] & x_1^* [(1 - \mu_m)\hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha]
\end{bmatrix} \\
&\quad - r \begin{bmatrix}
\hat{\alpha} w_1 (\hat{x}_3 - \mu_m) & -\hat{\alpha} w_3 (\hat{x}_1 - \mu_m) \\
-\hat{\beta} w_1 (\hat{x}_3 - \mu_m) & \hat{\beta} w_3 (\hat{x}_1 - \mu_m)
\end{bmatrix} \\
&\quad + r^2 (\alpha \hat{x}_3 + \beta \hat{x}_1) \begin{bmatrix}
\hat{\alpha} x_3^* & -\hat{\alpha} x_1^* \\
-\hat{\beta} x_3^* & \hat{\beta} x_1^*
\end{bmatrix}.
\end{align*}
\]

Let \( M(1) = \det(w^*\mathbf{L}_{\text{ex}} - w^*\mathbf{I}) \). Then in order to compute \( M(1) \) we multiply the first column of \( M(1) \) by \( x_1^* \), the second column by \( x_3^* \) to get \( M(1) = \frac{1}{x_1^* x_3^*} \det(B) \) where \( B \) is the following matrix:

\[
\begin{align*}
B &= \begin{bmatrix}
[(1 - \mu_m)^2 w_1 \hat{w}_1 + \mu_m^2 w_1 \hat{w}_3] x_1^* - w^* x_1^* & \mu_m(1 - \mu_m) w_3 (\hat{w}_1 + \hat{w}_3) x_3^* \\
\mu_m(1 - \mu_m) w_1 (\hat{w}_1 + \hat{w}_3) x_1^* & [(1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1] x_3^* - w^* x_3^*
\end{bmatrix} \\
&\quad - r \begin{bmatrix}
x_3^* [(1 - \mu_m)\hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta] & -x_1^* [(1 - \mu_m)\hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta] \\
-x_3^* [(1 - \mu_m)\hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha] & x_1^* [(1 - \mu_m)\hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha]
\end{bmatrix} \\
&\quad - r \begin{bmatrix}
\hat{\alpha} w_1 (\hat{x}_3 - \mu_m) & -\hat{\alpha} w_3 (\hat{x}_1 - \mu_m) \\
-\hat{\beta} w_1 (\hat{x}_3 - \mu_m) & \hat{\beta} w_3 (\hat{x}_1 - \mu_m)
\end{bmatrix} \\
&\quad + r^2 (\alpha \hat{x}_3 + \beta \hat{x}_1) \begin{bmatrix}
\hat{\alpha} x_3^* x_1^* & -\hat{\alpha} x_3^* x_1^* \\
-\hat{\beta} x_3^* x_1^* & \hat{\beta} x_3^* x_1^*
\end{bmatrix}.
\end{align*}
\]
We add the second column to the first and use the equilibrium equations for $w^*x_1^*$ and $w^*x_3^*$ from (25). The first column has a “constant” term, an “$r$” term and an “$r^2$” term. As we are adding the second column to the first, it is clear that the “$r^2$” term vanishes and the “$r$” term comes only from the second “$r$” matrix in (A18). Using the expression (A11) for $w^*$, the “constant” term in the first column, first row position is

$$\begin{align*}
[(1 - \mu_m)^2 w_1 \hat{\omega}_1 + \mu_m^2 w_1 \hat{\omega}_3] x_1^* + \mu_m (1 - \mu_m) w_3 (\hat{\omega}_1 + \hat{\omega}_3) x_3^* \\
- [(1 - \mu_M)^2 w_1 \hat{\omega}_1 + \mu_M^2 w_1 \hat{\omega}_3] x_1^* + \mu_M (1 - \mu_M) w_3 (\hat{\omega}_1 + \hat{\omega}_3) x_3^*. 
\end{align*}$$  \hfill (A19)

Noting that

$$\begin{align*}
(1 - \mu_m)^2 - (1 - \mu_M)^2 = (\mu_M - \mu_m)(2 - \mu_M - \mu_m), \\
\mu_m^2 - \mu_M^2 = (\mu_m - \mu_M)(\mu_M + \mu_m), \\
\mu_m (1 - \mu_m) - \mu_M (1 - \mu_M) = (\mu_m - \mu_M)(1 - \mu_M - \mu_m),
\end{align*}$$  \hfill (A20)

(A19) becomes

$$(\mu_m - \mu_M) \left[ -(2 - \mu_M - \mu_m) w_1 \hat{\omega}_1 x_1^* + (\mu_M + \mu_m) w_1 \hat{\omega}_3 x_1^* + (1 - \mu_M - \mu_m) w_3 (\hat{\omega}_1 + \hat{\omega}_3) x_3^* \right].$$  \hfill (A21)

Similarly the “constant” term in the first column and second row is

$$(\mu_m - \mu_M) \left[ -(2 - \mu_M - \mu_m) w_3 \hat{\omega}_3 x_3^* + (\mu_M + \mu_m) w_3 \hat{\omega}_1 x_3^* + (1 - \mu_M - \mu_m) w_1 (\hat{\omega}_1 + \hat{\omega}_3) x_1^* \right].$$  \hfill (A22)

The “$r$” terms are multiplied by $(-r)$ and in the first row are

$$\hat{\alpha} w_1 x_1^* (\hat{x}_3 - \mu_m) - \hat{\alpha} w_3 x_3^* (\hat{x}_1 - \mu_m),$$  \hfill (A23)

and in the second row

$$-\hat{\beta} w_1 x_1^* (\hat{x}_3 - \mu_m) + \hat{\beta} w_3 x_3^* (\hat{x}_1 - \mu_m).$$  \hfill (A24)

Now using equations (A6) for $\hat{x}_1$ and $\hat{x}_3$, we obtain

$$\begin{align*}
w_1 x_1^* (\hat{x}_3 - \mu_m) - w_3 x_3^* (\hat{x}_1 - \mu_m) &= w_1 x_1^* \frac{(1 - \mu_M) w_3 x_3^* + \mu_M w_1 x_1^*}{w_1 x_1^* + w_3 x_3^*} \\
&- \frac{w_3 x_3^* (1 - \mu_M) w_1 x_1^* + \mu_M w_3 x_3^*}{w_1 x_1^* + w_3 x_3^*} - \mu_m (w_1 x_1^* - w_3 x_3^*) \\
&= \frac{\mu_M}{w_1 x_1^* + w_3 x_3^*} \left( w_1 x_1^* + w_3 x_3^* \right)^2 - \mu_m (w_1 x_1^* - w_3 x_3^*).
\end{align*}$$  \hfill (A25)

We conclude that

$$w_1 x_1^* (\hat{x}_3 - \mu_m) - w_3 x_3^* (\hat{x}_1 - \mu_m) = (\mu_M - \mu_m) (w_1 x_1^* - w_3 x_3^*),$$  \hfill (A26)
Similarly (A24) reduces to

\[-\hat{\beta}(\mu_M - \mu_m)(w_1x_1^* - w_3x_3^*).\]

Hence we have a common factor of \((\mu_m - \mu_M)\) in the first column, and we can write

\[M(1) = \frac{(\mu_m - \mu_M)}{x_1^*x_3^*} \text{det}(A),\]

where \(\text{det}(A) = a_{11}a_{22} - a_{12}a_{21}\), with

\begin{align*}
a_{11} &= -(2 - \mu_M - \mu_m)w_1\hat{w}_1x_1^* + (\mu_M + \mu_m)w_1\hat{w}_3x_3^* \\
&\quad + (1 - \mu_M - \mu_m)w_3(\hat{w}_1 + \hat{w}_3)x_3^* - r\hat{\alpha}(w_3x_3^* - w_1x_1^*) \\
a_{12} &= \mu_m(1 - \mu_m)w_3(\hat{w}_1 + \hat{w}_3)x_3^* + rx_1^*x_3^*[(1 - \mu_m)\hat{w}_1\alpha - \mu_m\hat{w}_3\beta] \\
&\quad + r\hat{\alpha}w_3x_3^*(\hat{x}_1 - \mu_m) - r^2(\alpha\hat{x}_3 + \beta\hat{x}_1)\hat{\alpha}x_1^*x_3^* \\
a_{21} &= -(2 - \mu_M - \mu_m)w_3\hat{w}_3x_3^* + (\mu_M + \mu_m)w_3\hat{w}_1x_1^* \\
&\quad + (1 - \mu_M - \mu_m)w_1(\hat{w}_1 + \hat{w}_3)x_1^* + r\hat{\beta}(w_3x_3^* - w_1x_1^*) \\
a_{22} &= [(1 - \mu_m)^2w_3\hat{w}_3 + \mu_m^2w_3\hat{w}_1]x_3^* - w^*x_3^* - rx_1^*x_3^*[(1 - \mu_m)\hat{w}_3\beta - \mu_m\hat{w}_1\alpha] \\
&\quad - r\hat{\beta}w_3x_3^*(\hat{x}_1 - \mu_m) + r^2(\alpha\hat{x}_3 + \beta\hat{x}_1)\hat{\beta}x_1^*x_3^*.
\end{align*}

In \(\text{det}(A)\) we add the second row to the first using (A30) to obtain

\[a_{11} + a_{21} = -(2 - \mu_M - \mu_m)(w_1\hat{w}_1x_1^* + w_3\hat{w}_3x_3^*) + (\mu_M + \mu_m)(w_1\hat{w}_3x_3^* + w_3\hat{w}_1x_1^*) \\
+ (1 - \mu_M - \mu_m)(\hat{w}_1 + \hat{w}_3)(w_1x_1^* + w_3x_3^*) + r(\hat{\beta} - \hat{\alpha})(w_3x_3^* - w_1x_1^*).\]

The \((\mu_M + \mu_m)\) term vanishes, the constant term is \((\hat{w}_1 - \hat{w}_3)(w_3x_3^* - w_1x_1^*)\), and since \((\hat{\beta} - \hat{\alpha}) = (\hat{w}_3 - \hat{w}_1)\) we have

\[a_{11} + a_{21} = (1 - r)(\hat{w}_1 - \hat{w}_3)(w_3x_3^* - w_1x_1^*).\]

Observe that both \(a_{12}\) and \(a_{22}\) have a common factor of \(x_3^*\). Thus

\begin{align*}
\frac{a_{12} + a_{22}}{x_3^*} &= (1 - \mu_m)^2w_3\hat{w}_3 + \mu_m^2w_3\hat{w}_1 + \mu_m(1 - \mu_m)w_3(\hat{w}_1 + \hat{w}_3) - w^* \\
&\quad + rx_1^*(\hat{w}_1\alpha - \hat{w}_3\beta) + rw_3(\hat{\alpha} - \hat{\beta})(\hat{x}_1 - \mu_m) - r^2(\alpha\hat{x}_3 + \beta\hat{x}_1)(\hat{\alpha} - \hat{\beta})x_1^*,
\end{align*}

which we rewrite as

\begin{align*}
\frac{a_{12} + a_{22}}{x_3^*} &= T + (r - 1)x_1^*(\hat{w}_1\alpha - \hat{w}_3\beta) + (r - 1)w_3(\hat{\alpha} - \hat{\beta})(\hat{x}_1 - \mu_m) \\
&\quad - (r^2 - 1)(\alpha\hat{x}_3 + \beta\hat{x}_1)(\hat{\alpha} - \hat{\beta})x_1^*.
\end{align*}
where $T$ is independent of $r$.

Observe that when $r = 1$, from (A14) and (A15), $w\mathcal{L}$ and $\hat{w}\hat{\mathcal{L}}$ and their product $w^*\mathcal{L}_{ex}$ have the same structure, namely

$$w\mathcal{L} = \begin{bmatrix} \rho & \rho \\ \sigma & \sigma \end{bmatrix}, \quad \hat{w}\hat{\mathcal{L}} = \begin{bmatrix} \hat{\rho} & \hat{\rho} \\ \hat{\sigma} & \hat{\sigma} \end{bmatrix}, \quad w^*\mathcal{L}_{ex} = \begin{bmatrix} \Omega^* & \Omega^* \\ \nu^* & \nu^* \end{bmatrix}$$ (A35)

with

$$w = \rho + \sigma, \quad \hat{w} = \hat{\rho} + \hat{\sigma}, \quad w^* = \Omega^* + \nu^*.$$ 

Hence, when $r = 1$, $M(1) = \det(w^*\mathcal{L}_{ex} - w^*\mathcal{I})$ is in fact

$$M(1) = \begin{vmatrix} \Omega^* - w^* & \Omega \\ \nu^* & \nu^* - w^* \end{vmatrix}.$$ (A36)

Therefore, repeating the steps we used above to compute $M(1)$, we have

$$M(1) = \frac{1}{x_1^*x_3^*} \begin{vmatrix} (\Omega^* - w^*)x_1^* & \Omega^*x_3^* \\ \nu^*x_1^* & (\nu^* - w^*)x_3^* \end{vmatrix}. $$ (A37)

Since $x_1^* + x_3^* = 1$, adding the second column to the first gives

$$M(1) = \frac{1}{x_1^*x_3^*} \begin{vmatrix} \Omega^* - w^*x_1^* & \Omega^*x_3^* \\ \nu^* - w^*x_3^* & (\nu^* - w^*)x_3^* \end{vmatrix}. $$ (A38)

Adding the second row to the first gives

$$M(1) = \frac{1}{x_1^*x_3^*} \begin{vmatrix} \Omega^* + \nu^* - w^* & (\Omega^* + \nu^* - w^*)x_3^* \\ \nu^* - w^*x_3^* & (\nu^* - w^*)x_3^* \end{vmatrix}. $$ (A39)

But $w^* = \Omega^* + \nu^*$, so when $r = 1$ we should have $a_{11} + a_{21} = 0 = a_{12} + a_{22}$. Therefore in (A34) $T = 0$, and we conclude that

$$a_{12} + a_{22} = (1 - r)x_3^* [(r + 1)(\alpha \hat{x}_3 + \beta \hat{x}_1)x_1^*(\hat{\alpha} - \hat{\beta}) - x_1^*(\hat{\omega}_1 \alpha - \hat{\omega}_3 \beta) - \hat{w}_3(\hat{x}_1 - \mu_m)(\hat{\alpha} - \hat{\beta})].$$ (A40)

Thus $(a_{12} + a_{22})$ has also a factor of $(1 - r)$. In addition, both $(a_{12} + a_{22})$ and $a_{22}$ have a common factor of $x_3^*$, so in fact

$$M(1) = \frac{(\mu_m - \mu_M)(1 - r)}{x_1^*} \cdot \Delta,$$ (A41)
where $\Delta = d_{11}d_{22} - d_{12}d_{21}$ and

\begin{align*}
    d_{11} &= (\hat{w}_1 - \hat{w}_3)(w_3 x_3^* - w_1 x_1^*) \\
    d_{12} &= (r + 1)(\alpha \hat{x}_3 + \beta \hat{x}_1)x_1^*(\hat{\alpha} - \hat{\beta}) - x_1^*(\hat{w}_1 \alpha - \hat{w}_3 \beta) - w_3(\hat{x}_1 - \mu_m)(\hat{\alpha} - \hat{\beta}) \\
    d_{21} &= -(2 - \mu_M - \mu_m)w_3 \hat{w}_3 x_3^* + (\mu_M + \mu_m)w_3 \hat{w}_1 x_3^* + (1 - \mu_M - \mu_m)w_1(\hat{w}_1 + \hat{w}_3)x_1^* \\
        &\quad + r\hat{\beta}(w_3 x_3^* - w_1 x_1^*) \\
    d_{22} &= [(1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1] - w^* - r x_1^*[(1 - \mu_m)\hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha] \\
        &\quad - r\hat{\beta}w_3(\hat{x}_1 - \mu_m) + r^2(\alpha \hat{x}_3 + \beta \hat{x}_1)\hat{\beta}x_1^*.
\end{align*}

This proves equation (29). Thus the larger eigenvalue of $\mathbf{L}_{ex}$ is less than one if

\[
M(1) > 0 \quad \text{and} \quad M'(1) > 0,
\]

and greater than one if

\[
M(1) < 0.
\]