Part I: Some general properties of the transformation $\mathcal{L}$

We can gain some insight into the eigenvalues of $\mathcal{L}$ by the following observations.

**Observation I**

(i) \[
\det(\mathcal{L}_i^{\text{ex}}) = \frac{(1 - 2\mu_m)(1 - r)w_i^1 w_i^3}{(w^i)^2},
\]

(ii) \[
\det(\mathcal{L}_{\text{ex}}) = \frac{(1 - 2\mu_m)^k (1 - r)^k \prod_{i=1}^{k} w_i^1 w_i^3}{w^2}.
\]

The computation of $\det(\mathcal{L}_i^{\text{ex}})$ is straightforward and follows from the fact that $\mathcal{L}_i^{\text{ex}} = \mathcal{L}_k \cdot \mathcal{L}_{k-1} \cdot \cdots \cdot \mathcal{L}_1$.

Clearly when $\mu_m = \frac{1}{2}$, one of the eigenvalues of $\mathcal{L}_{\text{ex}}$ is zero. Also, if $0 \leq r < 1$ and $0 \leq \mu_m < \frac{1}{2}$, the two eigenvalues are positive for all $k$, whereas when $\frac{1}{2} < \mu_m \leq 1$ the two eigenvalues are positive for $k$ even and when $k$ is odd, one eigenvalue is positive, and the other negative.

**Observation II.**

When $r = 1$, the eigenvalues of each of the matrices $\mathcal{L}_i^{\text{ex}}$ and $\mathcal{L}_{\text{ex}}$ are zero and one.

In fact, when $r = 1$ the matrix $\mathcal{L}_i^{\text{ex}}$ has the form

\[
\mathcal{L}_i^{\text{ex}} = \frac{1}{a_i + b_i} \begin{bmatrix} a_i & a_i \\ b_i & b_i \end{bmatrix},
\]

with

\[
a_i = (1 - \mu_m)w_i^1 y_i^1 + \mu_m w_i^3 y_i^3 \\
b_i = \mu_m w_i^1 y_i^1 + (1 - \mu_m) w_i^3 y_i^3 \]

\[
a_i + b_i = w_i^1 y_i^1 + w_i^3 y_i^3 = w_i.
\]

Thus the two eigenvalues of $\mathcal{L}_i^{\text{ex}}$ are zero and one. Moreover it is easily seen that

\[
\mathcal{L}_i^{\text{ex}} \cdot \mathcal{L}_j^{\text{ex}} = \frac{1}{a_i + b_i} \begin{bmatrix} a_i & a_i \\ b_i & b_i \end{bmatrix} = \mathcal{L}_i^{\text{ex}}.
\]

Hence $\mathcal{L}_{\text{ex}} = \mathcal{L}_k^{\text{ex}} \cdot \mathcal{L}_{k-1}^{\text{ex}} \cdot \cdots \cdot \mathcal{L}_1^{\text{ex}}$ also has the form (A3), and so its eigenvalues are zero and one when $r = 1$. 
Part II: Proof of Equation (29)

Let \( \hat{\mathbf{x}} = (\hat{x}_1, 0, \hat{x}_3, 0) \) be such that \( \hat{\mathbf{x}} = T_1(\mathbf{x}^*) \), namely

\[
\begin{align*}
  w\hat{x}_1 &= (1 - \mu_M)w_1x_1^* + \mu_Mw_3x_3^*, \\
  w\hat{x}_3 &= (1 - \mu_M)w_3x_3^* + \mu_Mw_1x_1^*,
\end{align*}
\]

with

\[ w = w_1x_1^* + w_3x_3^*. \]  

Also let

\[ \hat{w} = \hat{w}_1\hat{x}_1 + \hat{w}_3\hat{x}_3. \]

Then the external stability of \( \mathbf{x}^* \) is determined by the largest positive eigenvalue of the matrix product \( \mathbf{L}_{ex} = \hat{\mathbf{L}} \cdot \mathbf{L} \) where

\[
\begin{align*}
  w\mathbf{L} &= \begin{bmatrix}
    (1 - \mu_m)w_1 - r[(1 - \mu_m)w_1 - \mu_m w_3]x_3^* & \mu_m w_3 + r[(1 - \mu_m)w_1 - \mu_m w_3]x_1^* \\
    \mu_m w_1 + r[(1 - \mu_m)w_3 - \mu_m w_1]x_3^* & (1 - \mu_m)w_3 - r[(1 - \mu_m)w_3 - \mu_m w_1]x_1^*
  \end{bmatrix},
\end{align*}
\]

and

\[
\begin{align*}
  \hat{w}\hat{\mathbf{L}} &= \begin{bmatrix}
    (1 - \mu_m)\hat{w}_1 - r[(1 - \mu_m)\hat{w}_1 - \mu_m \hat{w}_3]\hat{x}_3 & \mu_m \hat{w}_3 + r[(1 - \mu_m)\hat{w}_1 - \mu_m \hat{w}_3]\hat{x}_1 \\
    \mu_m \hat{w}_1 + r[(1 - \mu_m)\hat{w}_3 - \mu_m \hat{w}_1]\hat{x}_3 & (1 - \mu_m)\hat{w}_3 - r[(1 - \mu_m)\hat{w}_3 - \mu_m \hat{w}_1]\hat{x}_1
  \end{bmatrix}.
\end{align*}
\]

At equilibrium, we write \( \hat{w}\hat{\mathbf{L}} = w^* \), and following (26) we have

\[ w^* = \hat{w}_1 w_1 x_1^* + \hat{w}_3 w_3 x_3^* + \mu_M(\hat{w}_3 - \hat{w}_1)(w_1 x_1^* - w_3 x_3^*). \]

We can verify that

\[ \det(\mathbf{L}_{ex}) = \frac{(1 - 2\mu_m)^2(1 - r)^2w_1\hat{w}_1\hat{w}_3}{(w^*)^2}. \]  

Since \( \mathbf{L}_{ex} \) is a positive matrix, it has a positive eigenvalue, and (A12) ensures that \( \det(\mathbf{L}_{ex}) \) is positive when \( \mu_m \neq \frac{1}{2} \) and \( 0 \leq r < 1 \). If \( \mu_m = \frac{1}{2} \) or \( r = 1 \), \( \mathbf{L}_{ex} \) has a zero eigenvalue. Thus for all \( \mu_m \neq \frac{1}{2} \), \( \mathbf{L}_{ex} \) has one positive eigenvalue (the larger in magnitude) and another eigenvalue which is non-negative. The two eigenvalues are the roots of the characteristic polynomial \( C(z) = \det(\mathbf{L}_{ex} - z\mathbf{I}) \), or equivalently the roots of \( M(z) = \det(w^*\mathbf{L}_{ex} - w^*z\mathbf{I}) \).

The larger eigenvalue of \( \mathbf{L}_{ex} \) is less than one if \( M(1) > 0 \) and \( M'(1) > 0 \), and greater than one when \( M(1) < 0 \). \( M(1) \) has the useful representation

\[
M(1) = \frac{(\mu_m - \mu_M)(1 - r)}{x_1^*} \cdot \Delta,
\]  

where \( \Delta \) is defined as...
where $\Delta = \Delta(r, \mu_m) = d_{11}d_{22} - d_{12}d_{21}$ and $\Delta(r, \mu_m)$ is a bilinear function of $r$ and $\mu_m$ in $0 \leq r \leq 1$, $0 \leq \mu_m \leq 1$.

We begin by representing $w\mathbf{c}$ and $\hat{w}\hat{\mathbf{c}}$ as follows:

$$w\mathbf{c} = \begin{bmatrix} (1 - \mu_m)w_1 & \mu_m w_3 \\ \mu_m w_1 & (1 - \mu_m) w_3 \end{bmatrix} - r \begin{bmatrix} \alpha x_3^* & -\alpha x_1^* \\ -\beta x_3^* & \beta x_1^* \end{bmatrix}$$ (A14)

$$\hat{w}\hat{\mathbf{c}} = \begin{bmatrix} (1 - \mu_m)\hat{w}_1 & \mu_m \hat{w}_3 \\ \mu_m \hat{w}_1 & (1 - \mu_m) \hat{w}_3 \end{bmatrix} - r \begin{bmatrix} \hat{\alpha} \hat{x}_3 & -\hat{\alpha} \hat{x}_1 \\ -\hat{\beta} \hat{x}_3 & \hat{\beta} \hat{x}_1 \end{bmatrix},$$ (A15)

where

$$\alpha = w_1 - \mu_m(w_1 + w_3) \quad \beta = w_3 - \mu_m(w_1 + w_3)$$

$$\hat{\alpha} = \hat{w}_1 - \mu_m(\hat{w}_1 + \hat{w}_3) \quad \hat{\beta} = \hat{w}_3 - \mu_m(\hat{w}_1 + \hat{w}_3).$$ (A16)

We can now compute $w^*\mathbf{c}_{ex} = \hat{w}\hat{\mathbf{c}} \cdot w\mathbf{c}$ (using $w^* = w\hat{w}$) as follows:

$$w^*\mathbf{c}_{ex} = \begin{bmatrix} (1 - \mu_m)^2 w_1 \hat{w}_1 + \mu_m^2 w_1 \hat{w}_3 & \mu_m(1 - \mu_m) w_3(\hat{w}_1 + \hat{w}_3) \\ \mu_m(1 - \mu_m) w_1(\hat{w}_1 + \hat{w}_3) & (1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1 \end{bmatrix}
- r \begin{bmatrix} x_3^*[ (1 - \mu_m) \hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta ] & -x_1^*[ (1 - \mu_m) \hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta ] \\ -x_3^*[ (1 - \mu_m) \hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha ] & x_1^*[ (1 - \mu_m) \hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha ] \end{bmatrix}
- r \begin{bmatrix} \hat{\alpha} w_1(\hat{x}_3 - \mu_m) & -\hat{\alpha} w_3(\hat{x}_1 - \mu_m) \\ -\hat{\beta} w_1(\hat{x}_3 - \mu_m) & \hat{\beta} w_3(\hat{x}_1 - \mu_m) \end{bmatrix}
+ r^2(\alpha \hat{x}_3 + \beta \hat{x}_1) \begin{bmatrix} \hat{\alpha} x_3^* & -\hat{\alpha} x_1^* \\ -\hat{\beta} x_3^* & \hat{\beta} x_1^* \end{bmatrix}.$$ (A17)

Let $M(1) = \det(w^*\mathbf{c}_{ex} - w^*\mathbf{I})$. Then in order to compute $M(1)$ we multiply the first column of $M(1)$ by $x_1^*$, the second column by $x_3^*$ to get $M(1) = \frac{1}{x_1^* x_3^*} \det(B)$ where $B$ is the following matrix:

$$B = \begin{bmatrix} (1 - \mu_m)^2 w_1 \hat{w}_1 + \mu_m^2 w_1 \hat{w}_3 & x_3^* x_1^* - w^* x_1^* & \mu_m(1 - \mu_m) w_3(\hat{w}_1 + \hat{w}_3)x_3^* \\ \mu_m(1 - \mu_m) w_1(\hat{w}_1 + \hat{w}_3)x_1^* & (1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1 x_3 - w^* x_3^* \end{bmatrix}
- r \begin{bmatrix} x_3^*[ (1 - \mu_m) \hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta ] & -x_1^*[ (1 - \mu_m) \hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta ] \\ -x_3^*[ (1 - \mu_m) \hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha ] & x_1^*[ (1 - \mu_m) \hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha ] \end{bmatrix}
- r \begin{bmatrix} \hat{\alpha} w_1(\hat{x}_3 - \mu_m) & -\hat{\alpha} w_3(\hat{x}_1 - \mu_m) \\ -\hat{\beta} w_1(\hat{x}_3 - \mu_m) & \hat{\beta} w_3(\hat{x}_1 - \mu_m) \end{bmatrix}
+ r^2(\alpha \hat{x}_3 + \beta \hat{x}_1) \begin{bmatrix} \hat{\alpha} x_3 x_1 & -\hat{\alpha} x_3 x_1 \\ -\hat{\beta} x_3 x_1 & \hat{\beta} x_3 x_1 \end{bmatrix}.$$ (A18)
We add the second column to the first and use the equilibrium equations for \( w^* x_1^* \) and \( w^* x_3^* \) from (25). The first column has a “constant” term, an “\( r \)” term and an “\( r^2 \)” term. As we are adding the second column to the first, it is clear that the “\( r^2 \)” term vanishes and the “\( r \)” term comes only from the second “\( r \)” matrix in (A18). Using the expression (A11) for \( w^* \), the “constant” term in the first column, first row position is

\[
[(1 - \mu_m)^2 w_1 \bar{w}_1 + \mu_m^2 w_1 \bar{w}_3] x_1^* + \mu_m (1 - \mu_m) w_3 (\bar{w}_1 + \bar{w}_3) x_3^*
\]

\[-[(1 - \mu_M)^2 w_1 \bar{w}_1 + \mu_M^2 w_1 \bar{w}_3] x_1^* + \mu_M (1 - \mu_M) w_3 (\bar{w}_1 + \bar{w}_3) x_3^*.
\]

(A19)

Noting that

\[
(1 - \mu_m)^2 - (1 - \mu_M)^2 = (\mu_M - \mu_m)(2 - \mu_M - \mu_m),
\]

\[
\mu_m^2 - \mu_M^2 = (\mu_m - \mu_M)(\mu_M + \mu_m),
\]

\[
\mu_m (1 - \mu_m) - \mu_M (1 - \mu_M) = (\mu_m - \mu_M)(1 - \mu_M - \mu_m),
\]

(A20) becomes

\[
(\mu_m - \mu_M)[-(2 - \mu_M - \mu_m) w_1 \bar{w}_1 x_1^* + (\mu_M + \mu_m) w_1 \bar{w}_3 x_1^* + (1 - \mu_M - \mu_m) w_3 (\bar{w}_1 + \bar{w}_3) x_3^*].
\]

(A21)

Similarly the “constant” term in the first column and second row is

\[
(\mu_m - \mu_M)[-(2 - \mu_M - \mu_m) w_3 \bar{w}_3 x_3^* + (\mu_M + \mu_m) w_3 \bar{w}_1 x_3^* + (1 - \mu_M - \mu_m) w_1 (\bar{w}_1 + \bar{w}_3) x_1^*].
\]

(A22)

The “\( r \)” terms are multiplied by \((-r)\) and in the first row are

\[
\hat{\alpha} x_1 x_1^*(\hat{x}_3 - \mu_m) - \hat{\alpha} w_3 x_3^*(\hat{x}_1 - \mu_m),
\]

(A23)

and in the second row

\[
-\hat{\beta} w_1 x_1^*(\hat{x}_3 - \mu_m) + \hat{\beta} w_3 x_3^*(\hat{x}_1 - \mu_m).
\]

(A24)

Now using equations (A6) for \( \hat{x}_1 \) and \( \hat{x}_3 \), we obtain

\[
w_1 x_1^*(\hat{x}_3 - \mu_m) - w_3 x_3^*(\hat{x}_1 - \mu_m) = w_1 x_1^* \frac{(1 - \mu_M) w_3 x_3^* + \mu_M w_1 x_1^*}{w_1 x_1^* + w_3 x_3^*}
\]

\[-w_3 x_3^* \frac{(1 - \mu_M) w_1 x_1^* + \mu_M w_3 x_3^*}{w_1 x_1^* + w_3 x_3^*} - \mu_m (w_1 x_1^* - w_3 x_3^*)
\]

\[= \mu_M \frac{(w_1 x_1^*)^2 - (w_3 x_3^*)^2}{w_1 x_1^* + w_3 x_3^*} - \mu_m (w_1 x_1^* - w_3 x_3^*).
\]

(A25)

We conclude that

\[
w_1 x_1^*(\hat{x}_3 - \mu_m) - w_3 x_3^*(\hat{x}_1 - \mu_m) = (\mu_M - \mu_m)(w_1 x_1^* - w_3 x_3^*),
\]

(A26)
so (A23) becomes
\[ \hat{\alpha}(\mu_M - \mu_m)(w_1 x_1^* - w_3 x_3^*). \] (A27)

Similarly (A24) reduces to
\[ -\hat{\beta}(\mu_M - \mu_m)(w_1 x_1^* - w_3 x_3^*). \] (A28)

Hence we have a common factor of $(\mu_m - \mu_M)$ in the first column, and we can write
\[ M(1) = \frac{(\mu_m - \mu_M)}{x_1^* x_3^*} \det(A), \] (A29)

where $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, with
\[
\begin{align*}
a_{11} &= -(2 - \mu_M - \mu_m)w_1 \hat{w}_1 x_1^* + (\mu_M + \mu_m)w_1 \hat{w}_3 x_1^* \\
&\quad + (1 - \mu_M - \mu_m)w_3(\hat{w}_1 + \hat{w}_3) x_3^* - r\hat{\alpha}(w_3 x_3^* - w_1 x_1^*) \\
a_{12} &= \mu_m(1 - \mu_m)w_3(\hat{w}_1 + \hat{w}_3) x_3^* + r x_1^* x_3^* \left[ (1 - \mu_m)\hat{w}_1 \alpha - \mu_m \hat{w}_3 \beta \right] \\
&\quad + r\hat{\alpha}(w_3 x_3^* (\hat{x}_1 - \mu_m)) - r^2(\alpha \hat{x}_3 + \beta \hat{x}_1)\hat{x}_1^* x_3^* \\
a_{21} &= -(2 - \mu_M - \mu_m)w_3 \hat{w}_3 x_3^* + (\mu_M + \mu_m)w_3 \hat{w}_1 x_3^* \\
&\quad + (1 - \mu_M - \mu_m)w_1(\hat{w}_1 + \hat{w}_3) x_1^* + r\hat{\beta}(w_3 x_3^* - w_1 x_1^*) \\
a_{22} &= \left[ (1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1 \right] x_3^* - w_3 x_3^* - r x_1^* x_3^* \left[ (1 - \mu_m)\hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha \right] \\
&\quad - r\hat{\beta}w_3 x_3^*(\hat{x}_1 - \mu_m) + r^2(\alpha \hat{x}_3 + \beta \hat{x}_1)\hat{x}_1^* x_3^*. \\
\end{align*}
\] (A30)

In $\det(A)$ we add the second row to the first using (A30) to obtain
\[ a_{11} + a_{21} = -(2 - \mu_M - \mu_m)(w_1 \hat{w}_1 x_1^* + w_3 \hat{w}_3 x_3^*) + (\mu_M + \mu_m)(w_1 \hat{w}_3 x_3^* + w_3 \hat{w}_1 x_1^*) \\
+ (1 - \mu_M - \mu_m)(\hat{w}_1 + \hat{w}_3)(w_1 x_1^* + w_3 x_3^*) + r(\hat{\beta} - \hat{\alpha})(w_3 x_3^* - w_1 x_1^*). \] (A31)

The $(\mu_M + \mu_m)$ term vanishes, the constant term is $(\hat{w}_1 - \hat{w}_3)(w_3 x_3^* - w_1 x_1^*)$, and since $(\hat{\beta} - \hat{\alpha}) = (\hat{w}_3 - \hat{w}_1)$ we have
\[ a_{11} + a_{21} = (1 - r)(\hat{w}_1 - \hat{w}_3)(w_3 x_3^* - w_1 x_1^*). \] (A32)

Observe that both $a_{12}$ and $a_{22}$ have a common factor of $x_3^*$. Thus
\[ \frac{a_{12} + a_{22}}{x_3^*} = (1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1 + \mu_m(1 - \mu_m)w_3(\hat{w}_1 + \hat{w}_3) - w_3 x_3^* + r x_1^*(\hat{w}_1 \alpha - \hat{w}_3 \beta) + rw_3(\hat{\alpha} - \hat{\beta})(\hat{x}_1 - \mu_m) - r^2(\alpha \hat{x}_3 + \beta \hat{x}_1)(\hat{\alpha} - \hat{\beta})x_1^*, \] (A33)

which we rewrite as
\[ \frac{a_{12} + a_{22}}{x_3^*} = T + (r - 1)x_1^*(\hat{w}_1 \alpha - \hat{w}_3 \beta) + (r - 1)w_3(\hat{\alpha} - \hat{\beta})(\hat{x}_1 - \mu_m) \\
- (r^2 - 1)(\alpha \hat{x}_3 + \beta \hat{x}_1)(\hat{\alpha} - \hat{\beta})x_1^*, \] (A34)
where $T$ is independent of $r$.

Observe that when $r = 1$, from (A14) and (A15), $w \mathcal{L}$ and $\hat{w} \hat{\mathcal{L}}$ and their product $w^* \mathcal{L}_{ex}$ have the same structure, namely

$$w \mathcal{L} = \begin{bmatrix} \rho & \rho \\ \sigma & \sigma \end{bmatrix}, \quad \hat{w} \hat{\mathcal{L}} = \begin{bmatrix} \hat{\rho} & \hat{\rho} \\ \hat{\sigma} & \hat{\sigma} \end{bmatrix}, \quad w^* \mathcal{L}_{ex} = \begin{bmatrix} \Omega^* & \Omega^* \\ \nu^* & \nu^* \end{bmatrix}$$ (A35)

with

$$w = \rho + \sigma, \quad \hat{w} = \hat{\rho} + \hat{\sigma}, \quad w^* = \Omega^* + \nu^*.$$ Hence, when $r = 1$, $M(1) = \det(w^* \mathcal{L}_{ex} - w^* \mathcal{I})$ is in fact

$$M(1) = \begin{vmatrix} \Omega^* - w^* & \Omega \\ \nu^* & \nu^* - w^* \end{vmatrix}.$$ (A36)

Therefore, repeating the steps we used above to compute $M(1)$, we have

$$M(1) = \frac{1}{x_1^* x_3^*} \begin{vmatrix} (\Omega^* - w^*) x_1^* & \Omega^* x_3^* \\ \nu^* x_1^* & (\nu^* - w^*) x_3^* \end{vmatrix}.$$ (A37)

Since $x_1^* + x_3^* = 1$, adding the second column to the first gives

$$M(1) = \frac{1}{x_1^* x_3^*} \begin{vmatrix} \Omega^* - w^* x_1^* & \Omega^* x_3^* \\ \nu^* - w^* x_3^* & (\nu^* - w^*) x_3^* \end{vmatrix}.$$ (A38)

Adding the second row to the first gives

$$M(1) = \frac{1}{x_1^* x_3^*} \begin{vmatrix} \Omega^* + \nu^* - w^* & (\Omega^* + \nu^* - w^*) x_3^* \\ \nu^* - w^* x_3^* & (\nu^* - w^*) x_3^* \end{vmatrix}.$$ (A39)

But $w^* = \Omega^* + \nu^*$, so when $r = 1$ we should have $a_{11} + a_{21} = 0 = a_{12} + a_{22}$. Therefore in (A34) $T = 0$, and we conclude that

$$a_{12} + a_{22} = (1 - r) x_3^* \left[ (r + 1)(\alpha \hat{x}_3 + \beta \hat{x}_1) x_1^* (\hat{\alpha} - \hat{\beta}) - x_1^* (\hat{w}_1 \alpha - \hat{w}_3 \beta) - w_3 (\hat{x}_1 - \mu_m)(\hat{\alpha} - \hat{\beta}) \right].$$ (A40)

Thus $(a_{12} + a_{22})$ has also a factor of $(1 - r)$. In addition, both $(a_{12} + a_{22})$ and $a_{22}$ have a common factor of $x_3^*$, so in fact

$$M(1) = \frac{(\mu_m - \mu_M)(1 - r)}{x_1^*} \cdot \Delta,$$ (A41)
where $\Delta = d_{11}d_{22} - d_{12}d_{21}$ and

\[
\begin{align*}
d_{11} &= (\hat{w}_1 - \hat{w}_3)(w_3 x_3^* - w_1 x_1^*) \\
d_{12} &= (r + 1)(\alpha \hat{x}_3 + \beta \hat{x}_1)x_1^*(\hat{\alpha} - \hat{\beta}) - x_1^*(\hat{w}_1 \alpha - \hat{w}_3 \beta) - w_3(\hat{x}_1 - \mu_m)(\hat{\alpha} - \hat{\beta}) \\
d_{21} &= -(2 - \mu_M - \mu_m)w_3 \hat{w}_3 x_3^* + (\mu_M + \mu_m)w_3 \hat{w}_1 x_3^* + (1 - \mu_M - \mu_m)w_1(\hat{w}_1 + \hat{w}_3)x_1^* \\
&\quad + r\hat{\beta}(w_3 x_3^* - w_1 x_1^*) \\
d_{22} &= [(1 - \mu_m)^2 w_3 \hat{w}_3 + \mu_m^2 w_3 \hat{w}_1] - w^* - rx_1^*[ (1 - \mu_m) \hat{w}_3 \beta - \mu_m \hat{w}_1 \alpha] \\
&\quad - r\hat{\beta}w_3(\hat{x}_1 - \mu_m) + r^2(\alpha \hat{x}_3 + \beta \hat{x}_1)\hat{\beta}x_1^*. \tag{A42}
\end{align*}
\]

This proves equation (29). Thus the larger eigenvalue of $\mathcal{L}_{\text{ex}}$ is less than one if

\[
M(1) > 0 \quad \text{and} \quad M'(1) > 0, \tag{A43}
\]

and greater than one if

\[
M(1) < 0. \tag{A44}
\]
FILE S2: Proof that $\Delta = \Delta(r, \mu_m)$ in (29) (and A41) is a bilinear function of $r$ and $\mu_m$

$\Delta = d_{11}d_{22} - d_{12}d_{21}$. We will show that the coefficients of $r^2$ and of $\mu_m^2$ in the expansion of $\Delta$ are zero.

The coefficient of $r^2$ in $d_{11}d_{22}$ is

$$(\hat{w}_1 - \hat{w}_3)((w_3 x_3^* - w_1 x_1^*)(\alpha \hat{x}_3 + \beta \hat{x}_1)\hat{\beta} x_1^*), \quad (B1)$$

and since $\hat{\alpha} - \hat{\beta} = \hat{w}_1 - \hat{w}_3$ the coefficient of $r^2$ in $d_{12}d_{21}$ is

$$x_1^*(\alpha \hat{x}_3 + \beta \hat{x}_1)\hat{\beta}(w_3 x_3^* - w_1 x_1^*). \quad (B2)$$

Hence the coefficient of $r^2$ in $\Delta$ is zero.

Similarly, since $\alpha = w_1 - \mu_m(w_1 + w_3), \beta = w_3 - \mu_m(w_1 + w_3), \text{ and } \hat{\beta} = \hat{w}_3 - \mu_m(\hat{w}_1 + \hat{w}_3)$, the coefficient of $(\mu_m)^2$ in $d_{11}d_{22}$ is

$$(\hat{w}_1 - \hat{w}_3)((w_3 x_3^* - w_1 x_1^*) \times \left\{ w_3(\hat{w}_3 + \hat{w}_1) - rx_1^*(\hat{w}_1 + \hat{w}_3)(w_1 + w_3) - rw_3(\hat{w}_1 + \hat{w}_3) + r^2 x_1^*(\hat{w}_1 + \hat{w}_3)(w_1 + w_3) \right\}. \quad (B3)$$

(B3) reduces to

$$(\hat{w}_1 - \hat{w}_3)(\hat{w}_1 + \hat{w}_3)(w_3 x_3^* - w_1 x_1^*)(1 - r)[w_3 - rx_1^*(w_1 + w_3)]. \quad (B4)$$

The coefficient of $(\mu_m)^2$ in $d_{12}d_{21}$ is

$$\left[ -(1 + r)x_1^*(w_1 + w_3)(\hat{w}_1 - \hat{w}_3) + x_1^*(\hat{w}_1 - \hat{w}_3)(w_1 + w_3) + w_3(\hat{w}_1 - \hat{w}_3) \right] \times \left( w_3 x_3^* - w_1 x_1^* \right) \hat{\beta}(w_3 x_3^* - w_1 x_1^*), \quad (B5)$$

which simplifies to

$$(\hat{w}_1 - \hat{w}_3)(\hat{w}_1 + \hat{w}_3)(1 - r)(w_3 x_3^* - w_1 x_1^*)[w_3 - rx_1^*(w_1 + w_3)].$$

As (B4) and (B5) are equal, the coefficient of $(\mu_m)^2$ in $\Delta$ is indeed zero.
We first show, using (33), that at $x^*$ we have
\[
(w_1 - w_3)(w_1x_1^* - w_3x_3^*) > 0,
\] (C1)
and from (26) the mean fitness $w^*$ at $x^*$ is
\[
w^* = w_1w_3 + \mu_M(w_1 - w_3)(w_1x_1^* - w_3x_3^*),
\] (C2)
and is, therefore, an increasing function of $\mu_M$.

To show (C1), we compute $Q\left(\frac{w_3}{w_1}\right)$:
\[
Q\left(\frac{w_3}{w_1}\right) = (1 - \mu_M)\frac{w_3^2}{w_1^2}w_1 + \mu_M(w_3 - w_1)\frac{w_3}{w_1} - (1 - \mu_M)w_3
\] (C3)
\[
= \frac{w_3}{w_1}(w_3 - w_1).
\] (C4)
As $Q(0) < 0$ and $Q(\pm \infty) > 0$, when $w_3 > w_1$, we have $Q\left(\frac{w_3}{w_1}\right) > 0$ and $u^* = \frac{x_1^*}{x_3^*} > \frac{w_3}{w_1}$ or $w_3x_3^* > w_1x_1^*$. If $w_3 < w_1$, then $Q\left(\frac{w_3}{w_1}\right) < 0$ and $u^* = \frac{x_1^*}{x_3^*} > \frac{w_3}{w_1}$ or $w_3x_3^* < w_1x_1^*$. Hence $(w_1 - w_3)$ and $(w_1x_1^* - w_3x_3^*)$ have the same signs and (C1) follows. An alternative way to write the left side of (C1) is
\[
(w_1 - w_3)(w_1x_1^* - w_3x_3^*) = (w_1^2x_1^* + w_3^2x_3^* - w_1w_3),
\] (C5)
and $w_1^2x_1^* + w_3^2x_3^* > w_1w_3$. Again, using (C5) we get from (C2) that
\[
w^* = (1 - \mu_M)w_1w_3 + \mu_M(w_1^2x_1^* + w_3^2x_3^*).
\] (C6)
As $w_1^2x_1^* + w_3^2x_3^* > w_1w_3$, for all $0 \leq \mu_M \leq 1$
\[
w_1w_3 \leq w^* \leq w_1^2x_1^* + w_3^2x_3^*.
\] (C7)
Also, $w^* > w_1w_3$ when $\mu_M \neq 0$, and when $\mu_M \neq 1$, $w^* < w_1^2x_1^* + w_3^2x_3^*$.

We now look for the conditions under which $x^*$ is externally stable. These depend on the magnitude of the larger positive eigenvalue of $L_{ex}$, the product of (A9) and (A10). The determinant (A12) of this matrix in the symmetric case is given by
\[
\det(L_{ex}) = \frac{(1 - 2\mu_m)^2(1 - r)^2w_1^2w_3^2}{(w^*)^2}.
\] (C8)
As \( w^* \geq w_1 w_3 \), we conclude from (C8) that \( 0 \leq \det(\mathbf{L}_{\text{ex}}) \leq 1 \) for all possible values of \( r, \mu_m, \mu_M \). But \( \mathbf{L}_{\text{ex}} \) is a positive matrix and therefore has a positive eigenvalue. Hence the other eigenvalue of \( \mathbf{L}_{\text{ex}} \) is positive, except when \( \mu_m = \frac{1}{2} \) when it is zero. Their product is less than or equal to one, and also at most one eigenvalue is larger than one. In this case it is clear that the larger eigenvalue of \( \mathbf{L}_{\text{ex}} \) is greater than one if \( M(1) \) of (A13) is negative, and it is smaller than one if \( M(1) \) is positive.

Since \( \Delta(r, \mu_m) \) is a bilinear function of \( r \) and \( \mu_m \), the value of \( \Delta(r, \mu_m) \) is negative for \( 0 \leq r \leq 1, 0 \leq \mu_m \leq 1 \) if and only if the four “corner values”

\[
\Delta(0, 0), \quad \Delta(0, 1), \quad \Delta(1, 0), \quad \Delta(1, 1)
\]

are all negative.

In the case \( r = 0 \) we know, from (30), that the eigenvalues of \( \mathbf{L}_{\text{ex}} \), in the symmetric case, with \( \mu_m = 0 \) or \( \mu_m = 1 \) are

\[
\lambda_1^0 = \lambda_2^0 = \frac{w_1 w_3}{w^*} \quad \text{when } \mu_m = 0,
\]
\[
\lambda_1^1 = \frac{w_2^2}{w^*}, \quad \lambda_2^1 = \frac{w_3^2}{w^*} \quad \text{when } \mu_m = 1.
\]

When \( \mu_M > 0 \), (C6) and (C7) imply that \( w^* > w_1 w_3 \) and the two eigenvalues \( \lambda_1^0 \) and \( \lambda_2^0 \) are less than 1. Therefore when \( \mu_m = 0, \mu_M > 0 \), and \( r = 0 \), the equilibrium \( x^* \) is stable. From (29),

\[
M(1) = \frac{(\mu_m - \mu_M)(1 - r)}{x_1^*} \Delta(r, \mu_m).
\]

Hence \( M(1) \) should be positive when \( \mu_m = 0, \mu_M > 0 \), and \( r = 0 \). Thus \( \Delta(0, 0) \) is indeed negative.

Similarly, when \( \mu_M \neq 1, w_1^2 x_1^* + w_3^2 x_3^* > w^* \) by (C6) and (C7). Therefore when \( r = 0, \mu_m = 1, \) and \( 0 \leq \mu_M < 1 \) at least one of the eigenvalues \( \lambda_1^1 \) or \( \lambda_2^1 \) is larger than 1, and \( x^* \) is unstable. In this case \( M(1) \) should be negative when \( r = 0, \mu_m = 1, 0 \leq \mu_M < 1 \), which implies by (29) that \( \Delta(0, 1) \) is negative. The proof that \( \Delta(1, 0) \) and \( \Delta(1, 1) \) are negative is more complex.

We prove next that \( \Delta(1, 0) \) and \( \Delta(1, 1) \) are negative for all \( 0 < \mu_M < 1 \).

In the symmetric case where \( \hat{w}_1 = w_3, \hat{w}_3 = w_1 \) we have \( \hat{\alpha} = \beta, \hat{\beta} = \alpha \), where \( \alpha = \ldots \)
$w_1 - \mu_m (w_1 + w_3)$ and $\beta = w_3 - \mu_m (w_1 + w_3)$. Thus $\Delta (r, \mu_m) = d_{11} d_{22} - d_{12} d_{21}$, where

\[d_{11} = (w_1 - w_3) (w_1 x_1^* - w_3 x_3^*)\]
\[d_{12} = (1 + r) x_1^* (\alpha \hat{x}_3 + \beta \hat{x}_1) (\beta - \alpha) - x_1^* (w_3 \alpha - w_1 \beta) - w_3 (\hat{x}_1 - \mu_m) (\beta - \alpha)\]
\[d_{21} = -(2 - \mu_M - \mu_m) w_1 w_3 x_3^* + (\mu_M + \mu_m) w_3^2 x_3^* + (1 - \mu_M - \mu_m) w_1 (w_1 + w_3) x_1^* + r \alpha (w_3 x_3^* - w_1 x_1^*)\]
\[d_{22} = (1 - \mu_m)^2 w_1 w_3 + \mu_m^2 w_3^2 - w^* - r x_1^* [(1 - \mu_m) w_1 \beta - \mu_m w_3 \alpha] - r \alpha w_3 (\hat{x}_1 - \mu_m) + r^2 (\alpha \hat{x}_3 + \beta \hat{x}_1) \alpha x_1^*.
\]

We compute the $d_{ij}$’s of (C9) when $r = 1$ and $\mu_m = 0$, using the fact that in this case $\alpha = w_1$, $\beta = w_3$, and $\beta - \alpha = w_3 - w_1$, so that

\[d_{11} = (w_1 - w_3) (w_1 x_1^* - w_3 x_3^*)\]
\[d_{12} = (w_1 - w_3) [-x_1^* (w_1 \hat{x}_3 + w_3 \hat{x}_1) - \mu_M (w_1 x_1^* - w_3 x_3^*)]\]
\[d_{21} = -w_1 (w_1 x_1^* + w_3 x_3^*) - \mu_M (w_1 + w_3) (w_1 x_1^* - w_3 x_3^*)\]
\[d_{22} = -w_3 [w_1 x_1^* (1 - \mu_M) + \mu_M w_3 x_3^*].\]

When $w_1 > w_3$ we know that $(w_1 x_1^* - w_3 x_3^*) > 0$ by (C1); hence if $w_1 > w_3$ we have that $d_{11} > 0$, $d_{12} < 0$, $d_{21} < 0$, $d_{22} < 0$. Therefore clearly $\Delta (1, 0) = d_{11} d_{22} - d_{12} d_{21}$ is negative. Observe that in the symmetric case we can reorder the gametes from $AM Am aM am$ to $aM am AM Am$ without changing the results, as the mutation rates from $A$ to $a$ and from $a$ to $A$ are the same. This means that our results should be the same if we interchange $w_1 \leftrightarrow w_3$, $x_1^* \leftrightarrow x_3^*$, and $\hat{x}_1 \leftrightarrow \hat{x}_3$, in which case (C8) gives

\[M (1) = \frac{(\mu_m - \mu_M) (1 - r)}{x_3^*} \overline{\Delta} (r, \mu_m),\]

where $\overline{\Delta} (r, \mu_m) = \overline{d}_{11} \overline{d}_{22} - \overline{d}_{12} \overline{d}_{21}$ with the $\overline{d}_{ij}$’s obtained from the $d_{ij}$’s of (C9) by interchanging $x_1^* \leftrightarrow x_3^*$, $\hat{x}_1 \leftrightarrow \hat{x}_3$, and $w_1 \leftrightarrow w_3$.

From (C8) and (C11) we have

\[\frac{1}{x_1^*} \Delta (r, \mu_m) = \frac{1}{x_3^*} \overline{\Delta} (r, \mu_m),\]

and $\Delta (r, \mu_m)$ and $\Delta (r, \mu_m)$ have the same signs. Thus $\overline{\Delta} (1, 0) = \overline{d}_{11} \overline{d}_{22} - \overline{d}_{12} \overline{d}_{21}$, where from (C10),

\[\overline{d}_{11} = (w_3 - w_1) (w_3 x_3^* - w_1 x_1^*)\]
\[\overline{d}_{12} = (w_3 - w_1) [-x_3^* (w_3 \hat{x}_1 + w_1 \hat{x}_3) - \mu_M (w_3 x_3^* - w_1 x_1^*)]\]
\[\overline{d}_{21} = -w_3 (w_3 x_3^* + w_1 x_1^*) - \mu_M (w_3 + w_1) (w_3 x_3^* - w_1 x_1^*)\]
\[\overline{d}_{22} = -w_1 [(1 - \mu_M) w_3 x_3^* + \mu_M w_1 x_1^*].\]
But if \( w_3 > w_1 \) then \((w_3x_3^* - w_1x_1^*) > 0\), and consequently \(\bar{d}_{11} > 0\), whereas \(\bar{d}_{12} < 0, \bar{d}_{21} < 0, \) and \(\bar{d}_{22} < 0\). So \(\bar{\Delta}(1,0) = \bar{d}_{11}\bar{d}_{22} - \bar{d}_{12}\bar{d}_{21}\) is negative.

We thus proved that \(\Delta(1,0)\) is negative for all \(0 \leq \mu_M \leq 1\) when \(w_1 \neq w_3\).

When \(r = 1\) and \(\mu_m = 1\), we have \(\alpha = -w_3, \beta = -w_1, \) and \(\beta - \alpha = w_3 - w_1\). In this case the \(d_{ij}\)'s of (C9) reduce to

\[
\begin{align*}
  d_{11} &= (w_3 - w_1)(w_3x_3^* - w_1x_1^*) \\
  d_{12} &= (w_3 - w_1)[-2x_1^*(w_3\hat{x}_3 + w_1\hat{x}_1) + x_1^*(w_1 + w_3) + w_3\hat{x}_3] \\
  d_{21} &= w_1w_3(x_1^* - x_3^*) + \mu_M(w_1 + w_3)(w_3x_3^* - w_1x_1^*) \\
  w_{22} &= -w_1w_3^2x_3^* - \mu_M(w_3x_3^* - w_1x_1^*)[2w_3^2 - w_1(w_1x_1^* + w_3x_3^*)]
\end{align*}
\]

(C14)

where \(w = w_1x_1^* + w_3x_3^*\).

When \(w_3 > w_1\) we have \((w_3x_3^* - w_1x_1^*) > 0\) and also \(2w_3 > w_1(w_1x_1^* + w_3x_3^*)\). Therefore in this case \(d_{11} > 0\) and \(d_{22} < 0\). We will show that in this case \(d_{12}\) and \(d_{21}\) are both positive.

As \(w_3 > w_1\), \(d_{12}\) is positive if \(T\) is positive where

\[
T = -2x_1^*(w_1\hat{x}_1 + w_3\hat{x}_3) + x_1^*(w_1 + w_3) + w_3\hat{x}_3.
\]

(C15)

We use the equations for \(\hat{x}_1\) and \(\hat{x}_3\), namely,

\[
\begin{align*}
  w\hat{x}_1 &= (1 - \mu_M)w_1x_1^* + \mu_Mw_3x_3^* \\
  w\hat{x}_3 &= (1 - \mu_M)w_3x_3^* + \mu_Mw_1x_1^*,
\end{align*}
\]

(C16)

with \(w = w_1x_1^* + w_3x_3^*\). Using these equations and the fact that \(x_1^* + x_3^* = 1\), we get

\[
wT = (w_3x_3^* - w_1x_1^*)[w_3x_3^* + w_1x_1^*(1 - 2\mu_M) + w_3\mu_M(2x_1^* - 1)].
\]

(C17)

As \((w_3x_3^* - w_1x_1^*)\) is positive, \(T\) is positive if

\[
w_3x_3^* + w_1x_1^*(1 - 2\mu_M) + w_3\mu_M(2x_1^* - 1) > 0.
\]

(C18) is equivalent to

\[
w_3x_3^* + w_1x_1^* - w_3\mu_M + 2\mu_M(w_3 - w_1)x_1^* > 0.
\]

(C19)

As \(w_3 > w_1\), in order that (C19) holds it is sufficient that

\[
w_3x_3^* + w_1x_1^* > w_3\mu_M.
\]

(C20)
Write \( x_1^* = \frac{u^*}{1+u^*} \), \( x_3^* = \frac{1}{1+u^*} \) where \( \overline{Q}(u^*) = 0 \) and \( \overline{Q}(u) \) is given in (34). (C20) is thus equivalent to

\[
\begin{align*}
  w_3 + w_1 u^* > w_3 \mu_M (1 + u^*) \\
\end{align*}
\]

or

\[
\begin{align*}
  u^* (w_3 \mu_M - w_1) < w_3 (1 - \mu_M) .
\end{align*}
\]

When \( w_3 \mu_M < w_1 \), (C22) clearly holds. When \( w_3 \mu_M > w_1 \), then (C22) is true if and only if \( u^* \) satisfies

\[
\begin{align*}
  u^* < \frac{w_3 (1 - \mu_M)}{w_3 \mu_M - w_1} = t ,
\end{align*}
\]

say. Because of the properties of the quadratic equation \( \overline{Q}(u) = 0 \), \( u^* < t \) if and only if \( \overline{Q}(t) > 0 \). But from (34),

\[
\begin{align*}
  \overline{Q}(t) &= (1 - \mu_M) w_1 \frac{w_3^2 (1 - \mu_M)^2}{(w_3 \mu_M - w_1)^2} + \mu_M (w_3 - w_1) \frac{w_3 (1 - \mu_M)}{w_3 \mu_M - w_1} - (1 - \mu_M) w_3 .
\end{align*}
\]

As \( \mu_M < 1 \), the sign of \( \overline{Q}(t) \) coincides with the sign of

\[
\begin{align*}
  (1 - \mu_M)^2 w_1 w_3 + \mu_M (w_3 - w_1) (w_3 \mu_M - w_1) - (w_3 \mu_M - w_1)^2 .
\end{align*}
\]

But (C25) is equal to

\[
\begin{align*}
  (1 - \mu_M) w_1 (w_3 - w_1) ,
\end{align*}
\]

which is positive since \( w_3 > w_1 \) and \( 0 < \mu_M < 1 \). So indeed \( d_{12} > 0 \) when \( w_3 > w_1 \).

We now show that \( d_{21} \) is also positive when \( w_3 > w_1 \). Observe from (34) that \( \overline{Q}(1) = (1 - 2\mu_M) (w_1 - w_3) \). Therefore when \( w_3 > w_1 \) and \( 0 < \mu_M < \frac{1}{2} \), \( \overline{Q}(1) \) is negative, and so, based on the properties of \( \overline{Q}(u) = 0 \), we have \( u^* = \frac{x_1^*}{x_3^*} > 1 \) or \( x_1^* > x_3^* \). Hence, if \( w_3 > w_1 \) and also \( 0 < \mu_M < \frac{1}{2} \) then \( w_3 x_3^* > w_1 x_1^* \) and \( x_1^* > x_3^* \) implying that \( d_{21} \) is positive. We check the case when \( \frac{1}{2} < \mu_M < 1 \) and compute \( d_{21} \) using the representation \( x_1^* = \frac{u^*}{1+u^*} \), \( x_3^* = \frac{1}{1+u^*} \) with \( \overline{Q}(u^*) = 0 \). Indeed

\[
\begin{align*}
  (1 + u^*) d_{21} = w_1 w_3 (u^* - 1) + \mu_M (w_1 + w_3) (w_3 - w_1 u^*) .
\end{align*}
\]

Equivalently

\[
\begin{align*}
  (1 + u^*) d_{21} = w_1 u^* [w_3 - \mu_M (w_1 + w_3)] + w_3 [\mu_M (w_1 + w_3) - w_1] .
\end{align*}
\]

When \( \frac{1}{2} \leq \mu_M < 1 \), \( [\mu_M (w_1 + w_3) - w_1] > 0 \) as \( w_3 > w_1 \). If \( [w_3 - \mu_M (w_1 + w_3)] \geq 0 \) then clearly \( d_{21} \) is positive, and if \( [w_3 - \mu_M (w_1 + w_3)] < 0 \), \( d_{21} \) is positive provided
\[ u^* < \frac{w_3}{w_1} \cdot \frac{\mu_M(w_1 + w_3) - w_1}{\mu_M(w_1 + w_3) - w_3} = s, \]  
(C29)
say. (C29) is equivalent to \( \overline{Q}(s) > 0 \). Now

\[
\overline{Q}(s) = (1 - \mu_M)w_1 \frac{w_3^2}{w_1^2} \cdot \frac{[\mu_M(w_1 + w_3) - w_1]^2}{[\mu_M(w_1 + w_3) - w_3]^2}
+ \mu_M(w_3 - w_1) \frac{w_3}{w_1} \cdot \frac{\mu_M(w_1 + w_3) - w_1}{\mu_M(w_1 + w_3) - w_3} - (1 - \mu_M)w_3.
\]
(C30)
The sign of \( \overline{Q}(s) \) coincides with the sign of

\[
(1 - \mu_M)w_3 [\mu_M(w_1 + w_3) - w_1]^2 + \mu_M(w_3 - w_1) [\mu_M(w_1 + w_3) - w_1] [\mu_M(w_1 + w_3) - w_3]
- (1 - \mu_M)w_1 [\mu_M(w_1 + w_3) - w_3]^2.
\]
(C31)

(C31) is equal to

\[
(1 - \mu_M) \left\{ w_3 [\mu_M(w_1 + w_3) - w_1]^2 - w_1 [\mu_M(w_1 + w_3) - w_3]^2 \right\}
+ \mu_M(w_3 - w_1) [\mu_M(w_1 + w_3) - w_1] [\mu_M(w_1 + w_3) - w_3].
\]
(C32)
(C32) is equal to

\[
(1 - \mu_M) \left[ (w_3 - w_1) \mu_M^2(w_1 + w_3)^2 - w_1 w_3(w_3 - w_1) \right]
+ \mu_M(w_3 - w_1) [\mu_M^2(w_1 + w_3)^2 - \mu_M(w_1 + w_3)^2 + w_1 w_3].
\]
(C33)
As \( w_3 > w_1 \), (C33) is positive if

\[
(1 - \mu_M) [\mu_M^2(w_1 + w_3)^2 - w_1 w_3] + \mu_M [-\mu_M(1 - \mu_M)(w_1 + w_3)^2 + w_1 w_3]
\]
(C34)
is positive. Finally, (C34) equals

\[
w_1 w_3 (2\mu_M - 1),
\]
(C35)
which is positive if \( \frac{1}{2} < \mu_M < 1 \). If \( \mu_M = \frac{1}{2} \) then \([w_3 - \mu_M(w_1 + w_3)] = [\mu_M(w_1 + w_3) - w_1] = \frac{w_3 - w_1}{2}\), which is positive, and in view of (C28) \( d_{21} \) is positive. It follows that \( d_{21} \) is always positive when \( w_3 > w_1 \).

We conclude that when \( w_3 > w_1, d_{11} > 0, d_{12} > 0, d_{21} > 0, \) and \( d_{22} < 0, \) and in this case \( \Delta(1, 1) \) is negative.

Using the same symmetry argument we used for the case \( r = 1 \) and \( \mu_m = 0 \), we can show in the same way that \( \Delta(1, 1) \) is also negative when \( w_1 > w_3 \), making it negative whenever \( w_1 \neq w_3 \) for all \( 0 < \mu_M < 1 \).

We conclude that \( \Delta(r, \mu_m) < 0 \) for \( 0 \leq r \leq 1, \ 0 \leq \mu_m \leq 1. \)
FILE S4: Proof of Result 3

Recall from (26) that \( w^*(\mu_M) \) is given by

\[
w^*(\mu_M) = w_1 w_3 + \mu_M (w_1 - w_3) (w_1 x_1^* - w_3 x_3^*),
\]

and as \((w_1 - w_3)(w_1 x_1^* - w_3 x_3^*)\) is always positive, \( w^*(\mu_M) \) is an increasing function of \( \mu_M \) if

\[
a^* = a^*(\mu_M) = (w_1 - w_3)(w_1 x_1^* - w_3 x_3^*) \tag{D1}
\]
is an increasing function of \( \mu_M \).

The equilibrium frequencies \( x_1^* \) and \( x_3^* \) can be represented as

\[
x_1^* = u_1^* + u_1^*, \quad x_3^* = \frac{1}{1 + u^*}
\]
where \( u^* \) is the unique positive solution of \( Q(u) = 0 \)

\[
Q(u) = (1 - \mu_M) w_1 u^2 + \mu_M (w_3 - w_1) u - (1 - \mu_M) w_3, \tag{D2}
\]
or equivalently if \( \overline{Q}(u) = 0 \), where

\[
\overline{Q}(u) = w_1 u^2 + \frac{\mu_M}{1 - \mu_M} (w_3 - w_1) u - w_3. \tag{D3}
\]

Now \( a^* \), as a function of \( u^* \), is

\[
a^* = (w_1 - w_3) \left[ w_1 \frac{u^*}{1 + u^*} - w_3 \frac{1}{1 + u^*} \right] = (w_1 - w_3) \frac{w_1 u^* - w_3}{1 + u^*}. \tag{D4}
\]

The derivative of \( a^* \), with respect to \( \mu_M \), is

\[
\frac{da^*}{d\mu_M} = \frac{du^*}{d\mu_M} \frac{da^*}{du^*}, \tag{D6}
\]
and from (D5) we have

\[
\frac{da^*}{du^*} = \frac{(w_1 - w_3)(w_1 + w_3)}{(1 + u^*)^2}. \tag{D7}
\]

The derivative \( (u^*)' = \frac{du^*}{d\mu_M} \) can be computed by implicit differentiation of the equilibrium equation \( \overline{Q}(u^*) = 0 \) giving

\[
2 w_1 u^*(u^*)' + \frac{\mu_M}{1 - \mu_M} (w_3 - w_1)(u^*)' + \frac{(w_3 - w_1) u^*}{(1 - \mu_M)^2} = 0. \tag{D8}
\]

Hence

\[
\left[ 2 w_1 u^* - \frac{\mu_M}{1 - \mu_M} (w_1 - w_3) \right] (u^*)' = \frac{(w_1 - w_3) u^*}{(1 - \mu_M)^2}. \tag{D9}
\]
We will show that \[ [2w_1 u^* - \frac{\mu_M}{1-\mu_M} (w_1 - w_3)] \] is always positive. It is clearly positive when \( w_1 < w_3 \). In the case \( w_1 > w_3 \), it is positive if \( u^* > \frac{\mu_M}{1-\mu_M} \cdot \frac{w_1 - w_3}{2w_1} \). Observe that

\[
\bar{Q} \left( \frac{\mu_M}{1-\mu_M} \cdot \frac{w_1 - w_3}{2w_1} \right) = w_1 \left( \frac{\mu_M}{1-\mu_M} \right)^2 \frac{(w_1 - w_3)^2}{4w_1^2} - \left( \frac{\mu_M}{1-\mu_M} \right)^2 \frac{(w_1 - w_3)^2}{2w_1} - w_3 \tag{D10}
\]

\[
= - \frac{1}{4} \left( \frac{\mu_M}{1-\mu_M} \right)^2 \frac{(w_1 - w_3)^2}{w_1} - w_3 < 0. \tag{D11}
\]

As \( \bar{Q}(0) < 0 \) and \( \bar{Q}(\pm\infty) > 0 \), the unique positive root \( u^* \) of \( \bar{Q}(u) = 0 \) should satisfy

\[
u^* > \frac{\mu_M}{1-\mu_M} \cdot \frac{w_1 - w_3}{2w_1}. \tag{D12}\]

Thus \( [2w_1 u^* - \frac{\mu_M}{1-\mu_M} (w_1 - w_3)] \) is positive, and in view of (D9) and (D7) both \( \frac{da^*}{du^*} \) and \( \frac{du^*}{d\mu_M} \) have the sign of \( (w_1 - w_3) \), and consequently \( \frac{da^*}{d\mu_M} \) is positive. Thus we have proved \( w^*(\mu_M) \) is an increasing function of \( \mu_M \).
FILE S5: Proofs for Symmetric Case, n = 2; Result 4

Part I: Proof of (39); if \( r = 0 \), \((w_1 - w_3)(w_1 x_1^* - w_3 x_3^*) > 0\)

With \( M \) fixed and \( \tilde{x} = (x_1, x_3)^T \) we have

\[
\tilde{x}' = T_2 \circ T_2 \circ T_1 \circ T_1 \tilde{x}. \tag{E1}
\]

The total matrix \( T = T_2 \circ T_2 \circ T_1 \circ T_1 \) is written

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{E2}
\]

where \( T_1 \) and \( T_2 \) are given in (37) and

\[
A = w_1 w_3 \left[(1 - \mu_M)^2 w_3 + \mu_M^2 w_1\right] \left[(1 - \mu_M)^2 w_1 + \mu_M^2 w_3\right] + \mu_M^2 (1 - \mu_M)^2 w_1^2 (w_1 + w_3)^2
\]

\[
B = \mu_M (1 - \mu_M) w_3 (w_1 + w_3)^2 \left[(1 - \mu_M)^2 w_3 + \mu_M^2 w_1\right]
\]

\[
C = \mu_M (1 - \mu_M) w_1 (w_1 + w_3)^2 \left[(1 - \mu_M)^2 w_1 + \mu_M^2 w_3\right]
\]

\[
D = w_1 w_3 \left[(1 - \mu_M)^2 w_3 + \mu_M^2 w_1\right] \left[(1 - \mu_M)^2 w_1 + \mu_M^2 w_3\right] + \mu_M^2 (1 - \mu_M)^2 w_3^2 (w_1 + w_3)^2. \tag{E3}
\]

The mean fitness \( w = (A + C)x_1 + (B + D)x_3 \) is

\[
w = w_1 w_3 \left[(1 - \mu_M)^2 w_3 + \mu_M^2 w_1\right] \left[(1 - \mu_M)^2 w_1 + \mu_M^2 w_3\right] + \\
+ \mu_M^2 (1 - \mu_M)^2 (w_1 + w_3)^2 (w_1^2 x_1 + w_3^2 x_3) + \\
+ \mu_M (1 - \mu_M) (w_1 + w_3)^2 \left\{w_1 x_1 \left[(1 - \mu_M)^2 w_1 + \mu_M^2 w_3\right] + w_3 x_3 \left[(1 - \mu_M)^2 w_3 + \mu_M^2 w_1\right]\right\}. \tag{E4}
\]

We can simplify \( w \) to

\[
w = w_1^2 w_3^2 (1 - 2 \mu_M)^2 + \mu_M (1 - \mu_M) (w_1 + w_3)^2 \left[\mu_M w_1 w_3 + (1 - \mu_M) (w_1^2 x_1 + w_3^2 x_3)\right]. \tag{E5}
\]

Also

\[
w = w_1^2 w_3^2 - 4 \mu_M (1 - \mu_M) w_1^2 w_3^2 + \mu_M (1 - \mu_M) (w_1 + w_3)^2 w_1 w_3 + \\
+ \mu_M (1 - \mu_M) (w_1^2 x_1 + w_3^2 x_3 - w_1 w_3)(w_1 + w_3)^2 \tag{E6}
\]

\[
= w_1^2 w_3^2 + \mu_M (1 - \mu_M) w_1 w_3 (w_1 - w_3)^2 + \\
+ \mu_M (1 - \mu_M) (w_1^2 x_1 + w_3^2 x_3 - w_1 w_3)(w_1 + w_3)^2. \tag{E7}
\]

As \( w_1 x_1 + w_3 x_3 - w_1 w_3 = (w_1 - w_3)(w_1 x_1 - w_3 x_3) \) we can write

\[
w = w_1^2 w_3^2 + \mu_M (1 - \mu_M) (w_1 - w_3)^2 w_3 + \mu_M (1 - \mu_M)(w_1 - w_3)(w_1 x_1 - w_3 x_3)(w_1 + w_3)^2. \tag{E8}
\]
We will show that at equilibrium where $x = x^*$, $(w_1 - w_3)(w_1 x_1^* - w_3 x_3^*)$ is always positive so that $w \geq w_1^2 w_3^3$. In the equilibrium equation $Q(u) = 0$, where $x_1 = \frac{u}{1+u}$, $x_3 = \frac{1}{1+u}$, we have

$$Q(u) = \mu_M (1 - \mu_M) w_1 (w_1 + w_3) \left[ (1 - \mu_M)^2 w_1 + \mu_M^2 w_3 \right] u^2 + \mu_M (1 - \mu_M) (w_3 - w_1) (w_1 + w_3) u - \mu_M (1 - \mu_M) w_3 (w_1 + w_3) \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right].$$  \hspace{1cm} (E9)

Since $0 < \mu_M < 1$, $Q(u) = 0$ is equivalent to $\overline{Q}(u) = 0$ where

$$\overline{Q}(u) = w_1 \left[ (1 - \mu_M)^2 w_1 + \mu_M^2 w_3 \right] u^2 + \mu_M (1 - \mu_M) (w_3 - w_1) (w_3 + w_1) u - w_3 \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right].$$  \hspace{1cm} (E10)

But $\overline{Q}(0) < 0$ and $\overline{Q}(\pm \infty) > 0$; hence $\overline{Q}(u) = 0$ has one solution $u^* > 0$ and the other solution is negative. We compute

$$\overline{Q} \left( \frac{w_3}{w_1} \right) = \left[ (1 - \mu_M)^2 w_1 + \mu_M^2 w_3 \right] \frac{w_3^2}{w_1} + \mu_M (1 - \mu_M) (w_3^2 - w_1^2) \frac{w_3}{w_1} - w_3 \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right],$$  \hspace{1cm} (E11)

or

$$w_1 \overline{Q} \left( \frac{w_3}{w_1} \right) = w_3^2 \left[ (1 - \mu_M)^2 w_1 + \mu_M^2 w_3 \right] + \mu_M (1 - \mu_M) (w_3^2 - w_1^2) w_3 - w_1 w_3 \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right].$$  \hspace{1cm} (E12)

In fact

$$w_1 \overline{Q} \left( \frac{w_3}{w_1} \right) = w_3 (w_3 - w_1) \mu_M.$$  \hspace{1cm} (E13)

Thus the sign of $\overline{Q} \left( \frac{w_3}{w_1} \right)$ coincides with the sign of $(w_3 - w_1)$. Hence, if $w_3 > w_1$, $\overline{Q} \left( \frac{w_3}{w_1} \right)$ is positive and since $\overline{Q}(0) < 0$, $\overline{Q}(\infty) > 0$, we have $u^* = \frac{x_1^*}{x_3^*} < \frac{w_3}{w_1}$ or $w_3 x_3^* > w_1 x_1^*$. If $w_3 < w_1$ then $\overline{Q} \left( \frac{w_3}{w_1} \right) < 0$ and $u^* = \frac{x_1^*}{x_3^*} > \frac{w_3}{w_1}$ so that $w_3 x_3^* < w_1 x_1^*$. Therefore at equilibrium

$$(w_1 - w_3) (w_1 x_1^* - w_3 x_3^*) > 0.$$  \hspace{1cm} (E14)

**Part II: Proof of Result 4**

The mean fitness on the boundary where $M$ is fixed is $w^*(\mu_M)$ with

$$w^*(\mu_M) = w_1^2 w_3^2 + \mu_M (1 - \mu_M) w_1 w_3 (w_1 - w_3)^2 + \mu_M (1 - \mu_M) (w_1 + w_3)^2 (w_1 - w_3) (w_1 x_1^* - w_3 x_3^*).$$  \hspace{1cm} (E15)
Observe that $\mu_M(1 - \mu_M)$ is maximized when $\mu_M = \frac{1}{2}$ and also $(1 - \mu_M)(w_1 - w_3)(w_1 x_1^* - w_3 x_3^*) = a(\mu_M)$ is positive. Therefore $w^*(\mu_M)$ is maximized at $\mu_M = \frac{1}{2}$ if and only if $a(\mu_M)$ is maximized at $\mu_M = \frac{1}{2}$. When $\mu_M = \frac{1}{2}$, $x_1^* = x_3^* = \frac{1}{2}$ and $a \left( \frac{1}{2} \right) = \frac{1}{4}(w_1 - w_3)^2$. We will show that for any $0 \leq \mu_M \leq 1$, $a(\mu_M) \leq a \left( \frac{1}{2} \right)$. Recall that $x_1^* = \frac{u}{1+u}$, $x_3^* = \frac{1}{1+u}$, where $Q(u) = 0$. Hence

$$a(\mu_M) = (1 - \mu_M)(w_1 - w_3) \left( w_1 \frac{u}{1+u} - w_3 \frac{1}{1+u} \right) . \quad \text{(E16)}$$

Thus $a(\mu_M) \leq a \left( \frac{1}{2} \right)$ if and only if

$$ (1 - \mu_M)(w_1 - w_3)(w_1 u - w_3) \leq \frac{1}{4}(w_1 - w_3)^2(1 + u) , \quad \text{(E17)} $$

or if and only if

$$ u \left[ (1 - \mu_M)(w_1 - w_3)w_1 - \frac{1}{4}(w_1 - w_3)^2 \right] \leq (1 - \mu_M)(w_1 - w_3)w_3 + \frac{1}{4}(w_1 - w_3)^2 . \quad \text{(E18)} $$

We consider two cases.

**Case 1: $w_1 > w_3$**

In this case $(1 - \mu_M)(w_1 - w_3)w_3 + \frac{1}{4}(w_1 - w_3)^2 > 0$. If $(1 - \mu_M)(w_1 - w_3)w_1 - \frac{1}{4}(w_1 - w_3)^2 \leq 0$ then (E18) clearly holds. If $(1 - \mu_M)(w_1 - w_3)w_1 - \frac{1}{4}(w_1 - w_3)^2 > 0$ then (E18) holds if

$$ u \leq \frac{(1 - \mu_M)(w_1 - w_3)w_3 + \frac{1}{4}(w_1 - w_3)^2}{(1 - \mu_M)(w_1 - w_3)w_1 - \frac{1}{4}(w_1 - w_3)^2} = t , \quad \text{(E19)} $$

which is equivalent to $Q(t) > 0$ as $Q(0) < 0$ and $Q(\infty) > 0$.

**Case 2: $w_1 < w_3$**

In this case $(1 - \mu_M)(w_1 - w_3)w_1 - \frac{1}{4}(w_1 - w_3)^2 < 0$, and (E18) holds if

$$ u \geq \frac{(1 - \mu_M)(w_1 - w_3)w_3 + \frac{1}{4}(w_1 - w_3)^2}{(1 - \mu_M)(w_1 - w_3)w_1 - \frac{1}{4}(w_1 - w_3)^2} = t , \quad \text{(E20)} $$

which is equivalent to $Q(t) < 0$.

We will therefore show that

$$ w_1 > w_3 \rightarrow Q(t) > 0 \quad \text{(E21)} $$

$$ w_1 < w_3 \rightarrow Q(t) < 0 . \quad \text{(E22)} $$
This will prove that if \( w_1 \neq w_3 \) then \( a(\mu_M) \leq a(\frac{1}{2}) \). If \( w_1 \neq w_3 \),

\[
t = \frac{(1 - \mu_M)w_3 + \frac{1}{4}(w_1 - w_3)}{(1 - \mu_M)w_1 - \frac{1}{4}(w_1 - w_3)}.
\]

(E23)

Therefore \( \overline{Q}(t) \) equals

\[
w_1 \left[ (1 - \mu_M)^2 + \mu_M^2 w_3 \right] \left[ (1 - \mu_M)w_3 + \frac{1}{4}(w_1 - w_3) \right]^2
+ \mu_M (1 - \mu_M)(w_3^2 - w_1^2) \left[ (1 - \mu_M)w_3 + \frac{1}{4}(w_1 - w_3) \right] \left[ (1 - \mu_M)w_1 - \frac{1}{4}(w_1 - w_3) \right]
- w_3 \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right] \left[ (1 - \mu_M)w_1 - \frac{1}{4}(w_1 - w_3) \right].
\]

(E24)

Hence the sign of \( \overline{Q}(t) \) coincides with the sign of

\[
H = w_1 \left[ (1 - \mu_M)^2 w_1 + \mu_M^2 w_3 \right] \left[ (1 - \mu_M)w_3 + \frac{1}{4}(w_1 - w_3) \right]^2
+ \mu_M (1 - \mu_M)(w_3^2 - w_1^2) \left[ (1 - \mu_M)w_3 + \frac{1}{4}(w_1 - w_3) \right] \left[ (1 - \mu_M)w_1 - \frac{1}{4}(w_1 - w_3) \right]
- w_3 \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right] \left[ (1 - \mu_M)w_1 - \frac{1}{4}(w_1 - w_3) \right].
\]

(E25)

We compute \( H \).

\[
H = w_1 \left[ (1 - \mu_M)^2 w_1 + \mu_M^2 w_3 \right] \left[ (1 - \mu_M)^2 w_3 + \frac{1}{2}(1 - \mu_M)w_3(w_1 - w_3) + \frac{1}{16}(w_1 - w_3)^2 \right]
- w_3 \left[ (1 - \mu_M)^2 w_3 + \mu_M^2 w_1 \right] \left[ (1 - \mu_M)^2 w_1^2 - \frac{1}{2}(1 - \mu_M)w_1(w_1 - w_3) + \frac{1}{16}(w_1 - w_3)^2 \right]
+ \mu_M (1 - \mu_M)(w_3^2 - w_1^2) \left[ (1 - \mu_M)^2 w_1 w_3 + \frac{1}{4}(1 - \mu_M)(w_1 - w_3)^2 - \frac{1}{16}(w_1 - w_3)^2 \right].
\]

(E26)

\( H \) equals

\[
(1 - \mu_M)^4 w_1^2 w_3^2 + \frac{1}{2}(1 - \mu_M)^3 w_1^2 w_3^2(w_1 - w_3) + \frac{1}{16}(1 - \mu_M)^2 w_1^2(w_1 - w_3)^2
- (1 - \mu_M)^4 w_3^2 + \frac{1}{2}(1 - \mu_M)^3 w_1^2 w_3^2(w_1 - w_3) - \frac{1}{16}(1 - \mu_M)^2 w_3^2(w_1 - w_3)^2
+ \mu_M^2 (1 - \mu_M)^2 w_1 w_3^3 + \frac{1}{2}\mu_M^2 (1 - \mu_M)w_1 w_3^2(w_1 - w_3) + \frac{1}{16} w_1 w_3(w_1 - w_3)^2
- \mu_M^2 (1 - \mu_M)^2 w_3 w_1^3 + \frac{1}{2}\mu_M^2 (1 - \mu_M)w_1 w_3^2(w_1 - w_3) - \frac{1}{16} w_1 w_3(w_1 - w_3)^2
+ \mu_M (1 - \mu_M)^3 w_3^2(w_3^2 - w_1^2) + \frac{1}{4}\mu_M (1 - \mu_M)^2(w_3^2 - w_1^2)(w_1 - w_3)^2
- \frac{1}{16} \mu_M (1 - \mu_M)(w_3^2 - w_1^2)(w_1 - w_3)^2.
\]

(E27)

In fact \( H \) equals

\[
\frac{1 - \mu_M}{16} (1 - 2\mu_M)^2(w_1^2 - w_3^2)(w_1 - w_3)^2 + \frac{1 - \mu_M}{2} (1 - 2\mu_M)^2w_1 w_3(w_1^2 - w_3^2).
\]

(E28)

Thus

\[
H = \frac{1 - \mu_M}{16} (1 - 2\mu_M)^2(w_1 - w_3)(w_1 + w_3) \left[ (w_1 - w_3)^2 + 8w_1 w_3 \right].
\]

(E29)
If $\mu_M = \frac{1}{2}$ then $H = 0$; otherwise for all $0 < \mu_M < 1$ the sign of $H$ coincides with the sign of $(w_1 - w_3)$. Hence when $w_1 \neq w_3$, the sign of $\overline{Q}(T)$ coincides with the sign of $(w_1 - w_3)$, which proves that $a(\mu_M) \leq a\left(\frac{1}{2}\right)$ and $w^*(\mu_M)$ has its maximum value when $\mu_M = \frac{1}{2}$.

**Part III: Proof of Result 5**

We check the external stability of $\hat{x}^* = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ when $r = 0 \mu_M = \frac{1}{2}$ with $w^* = \left(\frac{w_1 + w_3}{2}\right)^4$. The eigenvalues of the linear approximation $L_{\text{ex}}$ solve the equation

$$M(\hat{z}) = \begin{bmatrix} \hat{A} - w^*z & \hat{B} \\ \hat{C} & \hat{D} - w^*z \end{bmatrix} = (w^*z)^2 - (w^*z) \left(\hat{A} + \hat{D}\right) + \hat{A}\hat{D} - \hat{B}\hat{C}, \quad (E30)$$

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are the values $A, B, C, D$ with $\mu_m$ instead of $\mu_M$.

As one of the eigenvalues is positive, by Perron-Frobenius and as $\det L_{\text{ex}} = \hat{A}\hat{D} - \hat{B}\hat{C} = (1 - 2\mu_m)^4 w_1^4 w_3^4$, the two eigenvalues are positive and the larger is less than one if and only if $M(1) > 0$.

$$M(1) = M(1; \mu_m) = (w^*)^2 - w^* \left(\hat{A} + \hat{D}\right) + \hat{A}\hat{D} - \hat{B}\hat{C} \quad (E31)$$

As $\hat{x}^*$ is the equilibrium with $\mu_M = \frac{1}{2}$, one of the eigenvalues is 1 when $\mu_m = \mu_M = \frac{1}{2}$. Therefore

$$M \left(1; \frac{1}{2}\right) = (w^*)^2 - w^* \left(\hat{A} + \hat{D}\right) + \hat{A}\hat{D} - \hat{B}\hat{C} = 0, \quad (E32)$$

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are the values of $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ when $\mu_m = \frac{1}{2}$. As $\hat{A}\hat{D} - \hat{B}\hat{C} = 0$, we have

$$M(1; \mu_m) = M(1; \mu_m) - M \left(1; \frac{1}{2}\right) = w^* \left[\left(\hat{A} + \hat{D}\right) - \left(\hat{A} + \hat{D}\right)\right] + \left(\hat{A}\hat{D} - \hat{B}\hat{C}\right). \quad (E33)$$

But as $M \left(1; \frac{1}{2}\right) = 0$, $w^* = \left(\hat{A} + \hat{D}\right)$ and

$$M(1; \mu_m) = w^* \left[w^* - \left(\hat{A} + \hat{D}\right)\right] + \left(\hat{A}\hat{D} - \hat{B}\hat{C}\right). \quad (E34)$$

Now $\left(\hat{A}\hat{D} - \hat{B}\hat{C}\right) = (1 - 2\mu_m)^4 w_1^4 w_3^4$.

$$\hat{A} + \hat{D} = 2w_1^2 w_3^3 \left[(1 - \mu_m)^4 + (\mu_m)^4\right] + 2\mu_m^2 (1 - \mu_m)^2 w_1 w_3 \left(w_1^2 + w_3^2\right) \quad (E35)$$

$$\hat{A} + \hat{D} = 2w_1 w_3 \left[w_1 w_3 \left[(1 - \mu_m)^4 + (\mu_m)^4\right] + \mu_m^2 (1 - \mu_m)^2 \left(w_1^2 + w_3^2\right)\right] \quad (E36)$$
\[ \tilde{A} + \tilde{D} = 2w_1 w_3 \times \]
\[ \times \left\{ w_1 w_3 \left[ (1 - \mu_m)^4 - 2\mu_m^2 (1 - \mu_m)^2 + (\mu_m)^4 \right] + \mu_m^2 (1 - \mu_m)^2 \left( w_1^2 + 2w_1 w_3 + w_3^2 \right)^2 \right\} \]
\[ + \mu_m^2 (1 - \mu_m)^2 \left( w_1^2 + w_3^2 \right) (w_1 + w_3)^2 \]
\[ \tilde{A} + \tilde{D} = 2w_1^2 w_3^2 \left[ (1 - \mu_m)^2 - (\mu_m)^2 \right]^2 + 2\mu_m^2 (1 - \mu_m)^2 (w_1 + w_3)^2 w_1 w_3 \]
\[ + \mu_m^2 (1 - \mu_m)^2 \left( w_1^2 + w_3^2 \right) (w_1 + w_3)^2 \]
\[ \tilde{A} + \tilde{D} = 2w_1^2 w_3^2 (1 - 2\mu_m)^2 + \mu_m^2 (1 - \mu_m)^2 (w_1 + w_3)^2 \left( w_1^2 + 2w_1 w_3 + w_3^2 \right)^2 . \]

Thus
\[ \tilde{A} + \tilde{D} = 2w_1^2 w_3^2 (1 - 2\mu_m)^2 + \mu_m^2 (1 - \mu_m)^2 (w_1 + w_4)^4 . \]

Therefore
\[ M(1; \mu_m) = w^* \left[ w^* - (\tilde{A} + \tilde{D}) \right] + \left( \tilde{A}D - \tilde{B}\tilde{C} \right) \]
\[ \text{is equal to} \]
\[ M(1; \mu_m) = (w^*)^2 - w^* \mu_m^2 (1 - \mu_m)^2 (w_1 + w_4)^4 - 2w^* w_1 w_3^2 (1 - 2\mu_m)^2 + (1 - 2\mu_m)^4 w_1^2 w_3^2 . \]

Hence
\[ M(1; \mu_m) = \left[ w^* - w_1^2 w_3^2 (1 - 2\mu_m)^2 \right]^2 - w^* \mu_m^2 (1 - \mu_m)^2 (w_1 + w_3)^4 . \]

Equivalently, as \( w^* = \frac{(w_1 + w_3)^4}{16} \) we get
\[ M(1; \mu_m) = (w^*)^2 \left\{ \left[ 1 - \frac{w_1^2 w_3^2}{w^*} (1 - 2\mu_m)^2 \right]^2 - 16\mu_m^2 (1 - \mu_m)^2 \right\} \]

using the mean inequality \( \sqrt{w_1 w_3} < \frac{w_1 + w_3}{2} \), with equality if and only if \( w_1 = w_3 \). So as \( w_1 \neq w_3 \), \( w_1^2 w_3^2 < \left( \frac{w_1 + w_3}{2} \right)^4 = w^* \). Therefore
\[ 1 - \frac{w_1^2 w_3^2}{w^*} (1 - 2\mu_m)^2 > 1 - (1 - 2\mu_m)^2 = 4\mu_m (1 - \mu_m) . \]

Hence for all \( 0 \leq \mu_m \leq 1 \) we get that \( M(1; \mu_m) > 0 \). We thus conclude that \( \bar{x}^* \) based on \( \mu_M = \frac{1}{2} \) is always externally stable when \( r = 0 \).

**Part IV: Proof of Result 5**

For general \( r \), we can reorganize \( M(1) \) to have the sign of the product \( K \cdot L \) where
\[ K = (1 - r)^2 (1 - 2\mu)^2 (w_1 - w_3)^2 \left[ (w_1 + w_3)^2 + 16w_1 w_3 \right] \]
and
\[ L = 2 (w_1 + w_3)^4 - \left\{ (1 - r)^2 (1 - 2\mu)^2 \left[ (w_1 + w_3)^4 + 16w_1^2 w_3^2 \right] \right\} . \]
Now

\[ L > 2 (w_1 + w_3)^4 - \left\{ (w_1 + w_3)^4 + 16 w_1^2 w_3^2 \right\} \]

\[ = (w_1 + w_3)^4 - 16 w_1^2 w_3^2 \]

\[ = \left[ (w_1 + w_3)^2 - 4 w_1 w_3 \right] \left[ (w_1 + w_3)^2 + 4 w_1 w_3 \right] \]

\[ = (w_1 - w_3)^2 \left[ (w_1 + w_3)^2 + 4 w_1 w_3 \right] \]

\[ > 0. \]

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Since \( K \) is obviously positive, we conclude that \( M(1) > 0 \) for all \( r \in (0, 1) \) when \( \mu_M = \frac{1}{2} \). Thus \( \mu_M = \frac{1}{2} \) is not invadable for all \( r \).